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6201 Fort Belvoir, VA 22060-6201

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Transformation and Self-Similarity Properties of Gamma and Weibull Fragment Size Distributions

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Culbert B. Laney

Prepared by:
Engility Corporation
8211 Terminal Road
Suite 1000
Lorton, VA 22079

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14. ABSTRACT
This paper describes properties of Gamma and Weibull size distributions, of the type commonly used for liquid and solid fragmentation. Starting with general three-parameter forms, this paper systematically derives, categorizes, and compares two-parameter Gamma and Weibull subsets. Based on comparisons with test data, rare variants may sometimes perform as well as, or better than, well-known variants. Gamma distributions have traditionally been used mostly for liquid fragmentation while Weibull distributions have traditionally been used mostly for solid fragmentation. Based on comparisons with test data, Gamma and Weibull distributions appear to be equally applicable to both liquid and solid fragmentation.

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UNIT CONVERSION TABLE

U.S. customary units to and from international units of measurement*

U.S. Customary Units	Multiply by → ← Divide by†		International Units
Length/Area/Volume			
inch (in)	2.54	$\times 10^{-2}$	meter (m)
foot (ft)	3.048	$\times 10^{-1}$	meter (m)
yard (yd)	9.144	$\times 10^{-1}$	meter (m)
mile (mi, international)	1.609 344	$\times 10^{3}$	meter (m)
mile (nmi, nautical, U.S.)	1.852	$\times 10^{3}$	meter (m)
barn (b)	1	$\times 10^{-28}$	square meter (m^2)
gallon (gal, U.S. liquid)	3.785 412	$\times 10^{-3}$	cubic meter (m^3)
cubic foot (ft^3)	2.831 685	$\times 10^{-2}$	cubic meter (m^3)
Mass/Density			
pound (lb)	4.535 924	$\times 10^{-1}$	kilogram (kg)
unified atomic mass unit (amu)	1.660 539	$\times 10^{-27}$	kilogram (kg)
pound-mass per cubic foot (lb ft^{-3})	1.601 846	$\times 10^{1}$	kilogram per cubic meter (kg m^{-3})
pound-force (lbf avoirdupois)	4.448 222		newton (N)
Energy/Work/Power			
electron volt (eV)	1.602 177	$\times 10^{-19}$	joule (J)
erg	1	$\times 10^{-7}$	joule (J)
kiloton (kt) (TNT equivalent)	4.184	$\times 10^{12}$	joule (J)
British thermal unit (Btu) (thermochemical)	1.054 350	$\times 10^{3}$	joule (J)
foot-pound-force (ft lbf)	1.355 818		joule (J)
calorie (cal) (thermochemical)	4.184		joule (J)
Pressure			
atmosphere (atm)	1.013 250	$\times 10^{5}$	pascal (Pa)
pound force per square inch (psi)	6.984 757	$\times 10^{3}$	pascal (Pa)
Temperature			
degree Fahrenheit (°F)	$[T(^\circ F) - 32]/1.8$		degree Celsius (°C)
degree Fahrenheit (°F)	$[T(^\circ F) + 459.67]/1.8$		kelvin (K)
Radiation			
curie (Ci) [activity of radionuclides]	3.7	$\times 10^{10}$	per second (s^{-1}) [becquerel (Bq)]
roentgen (R) [air exposure]	2.579 760	$\times 10^{-4}$	coulomb per kilogram (C kg^{-1})
rad [absorbed dose]	1	$\times 10^{-2}$	joule per kilogram (J kg^{-1}) [gray (Gy)]
rem [equivalent and effective dose]	1	$\times 10^{-2}$	joule per kilogram (J kg^{-1}) [sievert (Sv)]

*Specific details regarding the implementation of SI units may be viewed at http://www.bipm.org/en/si/.
†Multiply the U.S. customary unit by the factor to get the international unit. Divide the international unit by the factor to get the U.S. customary unit.

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Transformation and Self-Similarity Properties of
Gamma and Weibull Fragment Size Distributions

Culbert B. Laney[1]
Engility Corp., 8211 Terminal Rd, Lorton, VA 22079 U.S.A.

Abstract: *This paper describes properties of two-parameter Gamma and Weibull size distributions, of the type commonly used for liquid and solid fragmentation. Starting with general three-parameter forms, this paper systematically derives, categorizes, and compares logical variations on two-parameter Gamma and Weibull size distributions. Based on comparisons with test data, rare variants may sometimes perform as well as, or better than, well-known variants. Gamma distributions have traditionally been used for liquid fragmentation while Weibull distributions have traditionally been used for solid fragmentation. Based on comparisons with test data, Gamma and Weibull distributions appear to be equally applicable to both liquid and solid fragmentation.*

Keywords: fragment size distributions, aerosol size distributions, Gamma size distributions, Weibull size distributions, Rosin-Rammler size distributions, Nukiyama-Tanasawa size distributions, generalized Gamma size distributions, root normal size distributions, Mott-Linfoot size distributions, universal size distributions

1. Introduction

This paper describes the transformation and self-similarity properties of a wide range of two-parameter Gamma and Weibull size distributions, including those commonly used for liquid and solid fragmentation. Transformation properties have been previously explored in, e.g., Paloposki (1991). Self-similarity (normalization) properties have been previously explored in, e.g., Lee et. al. (2004). The current treatment expands on these earlier treatments.

Size distributions are commonly written in eight different forms. As defined here, the *transformation condition* requires that all eight forms have, at most, mild (integrable) singularities. It would not make physical sense for a size distribution to be well-behaved in one ordinary form but to experience severe (non-integrable) singularities in another ordinary form. As shown here, Gamma and Weibull size distributions, as classes, satisfy the transformation condition over a broad range of parameter space.

Bennett (1936) suggested expressing fragment size distributions in normalized forms, i.e., with the independent variable divided by a characteristic or average size. As a typical outcome, Levy et. al. (2010) observed: "astonishingly, a simple normalization of the x axis by the average fragment mass [or volume or diameter] gathers all the initially scattered data into a single curve."

Friedlander & Wang (1966) noted that such normalizations are simple examples of *self-preserving* or *self-similar* size distributions. Expanding on this observation, Spicer & Pratsinis (1996) defined self-similar size distributions as cases "when the steady-state size distributions

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[1] E-mail address: Bert.Laney@engilitycorp.com

scaled by the average particle volume [or diameter or mass] collapses onto a single size distribution."

As defined here, there are three *self-similarity conditions*:

1. When a size distribution is expressed in terms of a given average size, it should actually obtain that average size.
2. When a size distribution is expressed in terms of a given average size, and that average size changes, key free parameters in the size distribution should remain the same.
3. When a size distribution is expressed in terms of a given average size, and that average size changes, the size distribution should stay the same.

Many previous studies have concluded that different materials exposed to different conditions may still experience similar fragmentation outcomes. This suggests the existence of *standard* or *universal size distributions*. Well-known examples include Mott & Linfoot (1943), Marshall & Palmer (1948), and Simmons (1977). In essence, universal size distributions involve parameters that assume a small number of discrete values – as few as one or two – as opposed to an infinite number of continuous values.

The third self-similarity condition is an essential property of universal size distributions. A distribution is hardly "universal" if it changes when the arbitrary normalization changes. The second self-similarity condition is a desirable, but not essential, property of universal size distributions. Universals are typically associated with certain parameters. However, unless a universal obtains the second self-similarity condition, these parameters change when the arbitrary normalization changes.

2. Size Distributions

Let *D* be the fragment diameter and let *M* be the fragment mass. The eight most common ways of expressing size distributions are as follows:

> $F_M(D)$ [$F_M(M)$] is the mass fraction of fragments with diameters [masses] greater than or equal to *D* [*M*].
>
> $f_M(D)$ [$f_M(M)$] is the mass fraction of fragments with diameters [masses] in a range *dD* centered on *D* divided by *dD* [*dM* centered on *M* divided by *dM*]
>
> $F(D)$ [$F(M)$] is the number fraction of fragments with diameters [masses] greater than or equal to *D* [*M*]
>
> $f(D)$ [$f(M)$] is the number fraction of fragments with diameters [masses] in a range *dD* centered on *D* divided by *dD* [*dM* centered on *M* divided by *dM*]

This list excludes minor variations such as $1-F_M$ and the use of fragment volume instead of fragment mass.

Notice that F_M is monotone decreasing such that $F_M(0)=1$ and $F_M(\infty)=0$. In addition, f_M is always non-negative such that:

$$\int_0^\infty f_M(x)dx = 1 \tag{1a}$$

Similarly, F is monotone decreasing such that $F(0)=1$ and $F(\infty)=0$. In addition, f is always non-negative such that:

$$\int_0^\infty f(x)dx = 1 \tag{1b}$$

In standard probability theory, F is called a *complementary cumulative distribution function* (*CCDF*) and f is called a *probability density function* (*PDF*).

The eight forms defined above are related to each other by eight equations. The first of these equations is as follows:

$$M = s\rho D^m \tag{2}$$

where ρ is the density, s is a constant shape factor, and m is the spatial dimension ($1 \le m \le 3$).

For classic aerosols with nearly-spherical droplets, $m=3$ and $s=\pi/6$. Alternatively, Wittel et. al. (2006) found regular isotropic eggshell fragments have $m = 2$ and needle-like plate glass fragments have $m = 1.5$, which "implies that fragments have a self-affine character, meaning that the larger they are, the more elongated they get."

The next two transformation equations are as follows:

$$F_M(M) = F_M(D) \tag{3}$$

$$F(M) = F(D) \tag{4}$$

which come from the definitions of $F_M(D)$, $F_M(M)$, $F(D)$, and $F(M)$. The fourth transformation equation is as follows:

$$f_M(D) \sim D^m f(D) \tag{5a}$$

or equivalently:

$$f_M(M) \sim Mf(M) \tag{5b}$$

For example, see Brown (1989) and Brown & Wohletz (1995). The last four transformation equations are well-known and standard:

$$F_M(D) = -\int_D^\infty f_M(x)dx\,;\;\; f_M(D) = -\frac{dF_M}{dD} \tag{6}$$

$$F_M(M) = -\int_M^\infty f_M(x)dx\,;\;\; f_M(M) = -\frac{dF_M}{dM} \tag{7}$$

$$F(D) = -\int_D^\infty f(x)dx\,;\;\; f(D) = -\frac{dF}{dD} \tag{8}$$

$$F(M) = -\int_M^\infty f(x)dx\,;\;\; f(M) = -\frac{dF}{dM} \tag{9}$$

3. Average Sizes

Consider the following average diameters:

$$D_{avg} = \int_0^\infty Df(D)dD \tag{10}$$

$$D_{M\,avg} = \int_0^\infty D f_M(D)dD \tag{11}$$

Equation (10) and (11) are known as the *count mean diameter (CMD)* and the *mass mean diameter (MMD)*, respectively. The count mean diameter is also known as the *arithmetic mean diameter* or the *number mean diameter*. Similarly:

$$D'_{avg} = \frac{1}{\int_0^\infty \frac{f(D)}{D}dD} \tag{12}$$

$$D'_{M\,avg} = \frac{1}{\int_0^\infty \frac{f_M(D)}{D}dD} \tag{13}$$

Equation (13) is known as the *Sauter mean diameter (SMD)*. Finally, consider the following average masses:

$$M_{avg} = \int_0^\infty Mf(M)dM \tag{14}$$

$$M_{M\,avg} = \int_0^\infty M f_M(M)\,dM \tag{15}$$

For liquid atomization, it is common to use the following ratio:

$$R_M = \frac{D_{M\,avg}}{D'_{M\,avg}} \tag{16}$$

Notice that $D'_{M\,avg}$ emphasizes small fragments while $D_{M\,avg}$ emphasizes large fragments. As a result, $1 \le R_M \le \infty$ measures fragment size spread where $R_M = 1$ corresponds to the least possible spread (i.e. monodisperse) while $R_M = \infty$ corresponds to the greatest possible spread.

As an empirical observation, different size distributions are often almost the same when R_M is almost the same. Thus it is common to see aerosol size distributions specified exclusively in terms of R_M, e.g., Simmons (1977), Wu et. al. (1991), Chou & Faeth (1998), and Sallam et. al. (2006). Alternatively:

$$R \equiv \frac{D_{avg}}{D'_{avg}} \tag{17}$$

The observations made above about R_M apply equally to R.

When transforming size distributions between different forms, it is useful to have ratios such as the following:

$$Q = \frac{D_{M\,avg}}{D_{avg}} \tag{18}$$

:

$$S_M = \frac{M_{M\,avg}}{s\rho D^m_{M\,avg}} \tag{19a}$$

$$S = \frac{M_{avg}}{s\rho D^m_{avg}} \tag{19b}$$

Notice that S_M measures skewness, where $S_M = 1$ if the fragment size distribution is evenly balanced between small and large fragments, e.g., a uniform size distribution. Similar observations apply to *S*.

4. General Three-Parameter Form

4.1 Introduction

Assuming $n > 0$, Rosin & Rammler (1927, 1933) proposed the following general form for solid fragmentation:

$$f_M(D) \sim D^k \exp\left[-bD^n\right] \tag{20}$$

where k, n, and b are three free parameters. For a modern English language description of this distribution, see Stoyan (2013). Building on Rosin & Rammler (1927, 1933), Nukiyama & Tanasawa (1938a, b) proposed the following general form for liquid fragmentation:

$$f(D) \sim D^l \exp\left[-bD^n\right] \tag{21}$$

where l, n, and b are three free parameters. For a modern English language description of this distribution, see Hiroyasu (2006). By Equation (5), Equations (20) and (21) are identical if:

$$k = l + m \tag{22}$$

Bennett (1936) suggested dividing D by a reference diameter, e.g., the 63.2%-quantile. Using D_{avg} as the reference diameter, the final general three-parameter form is as follows:

$$f(D) = \frac{1}{AD_{avg}} \left(\frac{D}{D_{avg}} \right)^l \exp\left[-b \left(\frac{D}{D_{avg}} \right)^n \right] \tag{23}$$

Notice that Equation (23) appears to have five free parameters, namely, A, l, n, b, and D_{avg}. However, if Equation (23) obeys Equation (1) and obtains the correct D_{avg} per the first self-similarity condition, the number of free parameters is reduced from five to three.

Equations (20), (21), and (23) were originally derived empirically via fits to experimental data. More recently, Dumouchel (2006, 2009) showed that these equations can be derived theoretically using a maximum entropy approach; the given proof generalizes earlier work, e.g., Griffith (1943), Li & Tankin (1987), Cousin et. al. (1996).

Besides being called Rosin-Rammler and Nukiyama-Tanasawa size distributions, Equations (20), (21), and (23) are sometimes called *generalized Gamma size distributions* after Stacy (1962); see, e.g., Lushnikov (2010), Dumouchel (2009), Dumouchel et. al. (2012). Less commonly, Equations (20), (21), and (23) are called Fréchet (1927) size distributions; see, e.g., Vázquez & Gañán-Calvo (2010). In fact, Crooks (2010) cataloged over 50 different names for closely-related distributions, all described as special cases of Amoroso (1925) distributions. With so many names in play, Stoyan (2013) reasonably suggested that "a neutral technical name like 'powered exponential distribution' might be more suitable." However, this risks adding yet another name to an already long list.

4.2 Transformation Expressions

Using the expressions from Section 2, Equation (23) can be recast into eight equivalent forms; see Table 1. In Table 1, l, n, b, d, A, and B are parameters, m is the spatial dimension defined by Equation (2), $\Gamma(x)$ is the gamma function, and $\Gamma(s,x)$ is the upper incomplete gamma function.

Table 1a. Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions expressed in terms of *F* and *f*.

$F(D)=\dfrac{\Gamma\left[\dfrac{l+1}{n},b\left(\dfrac{D}{D_{avg}}\right)^{n}\right]}{\Gamma\left(\dfrac{l+1}{n}\right)}$	$F(M)=\dfrac{\Gamma\left[\dfrac{l+1}{n},d\left(\dfrac{M}{M_{avg}}\right)^{\frac{n}{m}}\right]}{\Gamma\left(\dfrac{l+1}{n}\right)}$
$f(D)=\dfrac{1}{AD_{avg}}\left(\dfrac{D}{D_{avg}}\right)^{l}\exp\left[-b\left(\dfrac{D}{D_{avg}}\right)^{n}\right]$	$f(M)=\dfrac{S}{mAM_{avg}}\left(\dfrac{SM}{M_{avg}}\right)^{\frac{l+1}{m}-1}\exp\left[-d\left(\dfrac{M}{M_{avg}}\right)^{\frac{n}{m}}\right]$

Table 1b. Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions expressed in terms of F_M and f_M.

$F_M(D)=\dfrac{\Gamma\left[\dfrac{l+m+1}{n},b\left(\dfrac{D}{D_{avg}}\right)^{n}\right]}{\Gamma\left(\dfrac{l+m+1}{n}\right)}$	$F_M(M)=\dfrac{\Gamma\left[\dfrac{l+m+1}{n},d\left(\dfrac{M}{M_{avg}}\right)^{\frac{n}{m}}\right]}{\Gamma\left(\dfrac{l+m+1}{n}\right)}$
$f_M(D)=\dfrac{1}{BD_{avg}}\left(\dfrac{D}{D_{avg}}\right)^{l+m}\exp\left[-b\left(\dfrac{D}{D_{avg}}\right)^{n}\right]$	$f_M(M)=\dfrac{S}{mBM_{avg}}\left(\dfrac{SM}{M_{avg}}\right)^{\frac{l+1}{m}}\exp\left[-d\left(\dfrac{M}{M_{avg}}\right)^{\frac{n}{m}}\right]$

The expressions in Table 1 are valid if $s>0$ and $x>0$ in $\Gamma(s,x)$. The first condition is true if:

$$\frac{l+1}{n}>0\ ;\ \frac{l+m+1}{n}>0$$

Equivalently:

$$l>-1 \text{ and } n>0 \tag{24a}$$

or:

$$l < -m - 1 \text{ and } n < 0 \tag{24b}$$

The second condition is true if:

$$b > 0 \,;\ d > 0 \tag{25}$$

Paloposki (1991) first obtained these results for the special case $m = 3$.

Notice that Equation (24) changes if the choice of the average fragment size changes. This is discussed further in Section 4.3. In addition, notice that Equation (24) allows unphysical size distributions with infinite R or R_M. Section 4.4 derives more restrictive conditions that avoid this.

Notice that if n is negative, which is rare, these are sometimes called *negative* or *inverse generalized Gamma distributions*, Twomey (1977), Deepak & Box (1982), Kondratyev et. al. (2006). The best known negative generalized Gamma (a.k.a., Rosin-Rammler or Nukiyama-Tanasawa) size distribution is due to Griffith (1943); see also Tishkoff & Law (1977) and Grady & Kipp (1987).

4.3 Self-Similarity Condition 1

The first self-similarity condition requires that Equation (23) obtain the correct D_{avg} and M_{avg}. Let:

$$W_i = \int_0^\infty x^{l+i} \exp\left[-bx^n\right] dx \tag{26}$$

Assuming $b > 0$ and $(l+i+1)/n > 0$:

$$W_i = \frac{\Gamma\left(\dfrac{l+i+1}{n}\right)}{|n| b^{\frac{l+i+1}{n}}} \tag{27}$$

Table 2 shows the parameter settings required to ensure the first self-similarity condition.

Table 2. Relationships among parameters for Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions which ensure the correct D_{avg} and M_{avg}.

b	$W_0 = W_1$	$\Gamma\left(\frac{l+2}{n}\right)^n \Big/ \Gamma\left(\frac{l+1}{n}\right)^n$
d	$bS^{n/m}$	$\Gamma\left(\frac{l+m+1}{n}\right)^{\frac{n}{m}} \Big/ \Gamma\left(\frac{l+1}{n}\right)^{\frac{n}{m}}$
S	$\frac{W_m}{W_0}$	$b^{-\frac{m}{n}}\,\Gamma\left(\frac{l+m+1}{n}\right) \Big/ \Gamma\left(\frac{l+1}{n}\right)$
A	W_0	$\frac{1}{\lvert n\rvert}\, b^{-\frac{l+1}{n}}\,\Gamma\left(\frac{l+1}{n}\right)$
B	W_m	$\frac{1}{\lvert n\rvert}\, b^{-\frac{l+m+1}{n}}\,\Gamma\left(\frac{l+m+1}{n}\right)$

Notice that, as expected, all of the expressions in Table 2 are valid if Equation (24) is true.

Suppose $D_{M\,avg}$ is used as the reference diameter instead of D_{avg}. Then Equation (23) becomes:

$$f(D) = \frac{1}{A_M D_{M\,avg}} \left(\frac{D}{D_{M\,avg}}\right)^{l_M} \exp\left[-b_M \left(\frac{D}{D_{M\,avg}}\right)^{n_M}\right] \qquad (28)$$

In this case, Table 3 replaces Table 2.

Table 3. Relationships among parameters for Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions which ensure the correct $D_{M\,avg}$ and $M_{M\,avg}$.

b_M	$W_m = W_{m+1}$	$\Gamma\left(\frac{l_M+m+2}{n_M}\right)^{n_M} \Big/ \Gamma\left(\frac{l_M+m+1}{n_M}\right)^{n_M}$
d_M	$b_M S_M^{n_M/m}$	$\Gamma\left(\frac{l_M+2m+1}{n_M}\right)^{\frac{n_M}{m}} \Big/ \Gamma\left(\frac{l_M+m+1}{n_M}\right)^{\frac{n_M}{m}}$
S_M	$\frac{W_{2m}}{W_m}$	$b_M^{-\frac{m}{n_M}}\,\Gamma\left(\frac{l_M+2m+1}{n_M}\right) \Big/ \Gamma\left(\frac{l_M+m+1}{n_M}\right)$
A_M	W_0	$\frac{1}{\lvert n_M\rvert}\, b_M^{-\frac{l_M+1}{n_M}}\,\Gamma\left(\frac{l_M+1}{n_M}\right)$
B_M	W_m	$\frac{1}{\lvert n_M\rvert}\, b_M^{-\frac{l_M+m+1}{n_M}}\,\Gamma\left(\frac{l_M+m+1}{n_M}\right)$

Notice that all the expressions in Table 3 are valid if:

$$\frac{l_M+1}{n_M}>0 \quad \text{and} \quad \frac{l_M+2m+1}{n_M}>0$$

Equivalently:

$$l_M>-1 \text{ and } n_M>0 \tag{29a}$$

or:

$$l_M<-2m-1 \text{ and } n_M<0 \tag{29b}$$

In other words, Equation (29) replaces Equation (24) when $D_{M\,avg}$ replaces D_{avg}.

Suppose D'_{avg} is used as the reference diameter instead of D_{avg}. Then Equation (23) becomes:

$$f(D)=\frac{1}{A'D'_{avg}}\left(\frac{D}{D'_{avg}}\right)^{l'}\exp\left[-b'\left(\frac{D}{D'_{avg}}\right)^{n'}\right] \tag{30}$$

In this case, Table 4 replaces Table 2.

Table 4. Relationships among parameters for Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions which ensure the correct D'_{avg} and M'_{avg}.

b'	$W_0=W_{-1}$	$\Gamma\left(\frac{l'+1}{n'}\right)^{n'} \Big/ \Gamma\left(\frac{l'}{n'}\right)^{n'}$
d'	$b'S'^{n'/m}$	$\Gamma\left(\frac{l'+1}{n'}\right)^{\frac{n'}{m}} \Big/ \Gamma\left(\frac{l'-m+1}{n'}\right)^{\frac{n'}{m}}$
S'	$\frac{W_0}{W_{-m}}$	$b'^{-\frac{m}{n'}}\,\Gamma\left(\frac{l'+1}{n'}\right) \Big/ \Gamma\left(\frac{l'-m+1}{n'}\right)$
A'	W_0	$\frac{1}{\lvert n'\rvert}\,b'^{-\frac{l'+1}{n'}}\,\Gamma\left(\frac{l'+1}{n'}\right)$
B'	W_m	$\frac{1}{\lvert n'\rvert}\,b'^{-\frac{l'+m+1}{n'}}\,\Gamma\left(\frac{l'+m+1}{n'}\right)$

Notice that all the expressions in Table 4 are valid if:

$$\frac{l'-m+1}{n'}>0 \quad \text{and} \quad \frac{l'+m+1}{n'}>0$$

Equivalently:

$$l' > m-1 \text{ and } n' > 0 \tag{31a}$$

or:

$$l' < -m-1 \text{ and } n' < 0 \tag{31b}$$

In other words, Equation (31) replaces Equation (24) when D'_{avg} replaces D_{avg}.

Suppose $D'_{M\,avg}$ is used as the reference diameter instead of D_{avg}. Then Equation (23) becomes:

$$f(D) = \frac{1}{A'_M D'_{M\,avg}} \left(\frac{D}{D'_{M\,avg}} \right)^{l'_M} \exp\left[-b'_M \left(\frac{D}{D'_{M\,avg}} \right)^{n'_M} \right] \tag{32}$$

In this case, Table 5 replaces Table 2.

Table 5. Relationships among parameters for Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions which ensure the correct $D'_{M\,avg}$ and $M'_{M\,avg}$.

b'_M	$W_m = W_{m-1}$	$\Gamma\left(\frac{l'_M + m + 1}{n'_M}\right)^{n'_M} \Big/ \Gamma\left(\frac{l'_M + m}{n'_M}\right)^{n'_M}$
d'_M	$b'_M S'^{n'_M/m}$	$\Gamma\left(\frac{l'_M + m + 1}{n'_M}\right)^{\frac{n'_M}{m}} \Big/ \Gamma\left(\frac{l'_M + 1}{n}\right)^{\frac{n'_M}{m}}$
S'_M	$\frac{W_m}{W_0}$	$b^{-\frac{m}{n'_M}} \Gamma\left(\frac{l'_M + m + 1}{n'_M}\right) \Big/ \Gamma\left(\frac{l'_M + 1}{n'_M}\right)$
A'_M	W_0	$\frac{1}{\lvert n'_M \rvert} b'^{-\frac{l'_M + 1}{n'_M}}_M \Gamma\left(\frac{l'_M + 1}{n'_M}\right)$
B'_M	W_m	$\frac{1}{\lvert n'_M \rvert} b'^{-\frac{l'_M + m + 1}{n'_M}}_M \Gamma\left(\frac{l'_M + m + 1}{n'_M}\right)$

Notice that all the expressions in Table 5 are valid if:

$$\frac{l'_M + 1}{n'_M} > 0 \quad \text{and} \quad \frac{l'_M + m + 1}{n'_M} > 0$$

Equivalently:

$$l'_M > -1 \text{ and } n_M > 0 \tag{33a}$$

or:

$$l'_M < -m-1 \text{ and } n'_M < 0 \tag{33b}$$

In other words, Equation (33) replaces Equation (24) when $D'_{M\,avg}$ replaces D_{avg}.

4.4 Self-Similarity Condition 2

The second self-similarity condition requires that changing the average fragment size does not change certain key parameters. For example, suppose the invariant parameters are *l* and *n*. In other words:

$$l = l_M = l' = l'_M$$
$$n = n_M = n' = n'_M$$

Table 6 gives expressions for ratios of averages for Equation (23), (28), (30), and (32).

Table 6. Ratios of averages for Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions assuming *l* and *n* are invariant.

Q	$\frac{W_0}{W_1}\frac{W_{m+1}}{W_m}$	$\dfrac{\Gamma\left(\frac{l+1}{n}\right)\Gamma\left(\frac{l+m+2}{n}\right)}{\Gamma\left(\frac{l+2}{n}\right)\Gamma\left(\frac{l+m+1}{n}\right)}$
R	$\frac{W_{-1}W_1}{W_0^2}$	$\dfrac{\Gamma\left(\frac{l+2}{n}\right)\Gamma\left(\frac{l}{n}\right)}{\Gamma\left(\frac{l+1}{n}\right)^2}$
R_M	$\frac{W_{m-1}W_{m+1}}{W_m^2}$	$\dfrac{\Gamma\left(\frac{l+m+2}{n}\right)\Gamma\left(\frac{l+m}{n}\right)}{\Gamma\left(\frac{l+m+1}{n}\right)^2}$

Notice that the expressions in Table 6 are valid if:

$$\frac{l}{n} > 0 \quad \text{and} \quad \frac{l+m+2}{n} > 0$$

Equivalently:

$$l > 0 \text{ and } n > 0 \tag{34a}$$

or:

$$l < -m - 2 \text{ and } n < 0 \tag{34b}$$

Where it is more restrictive, it is recommendation that Equation (34) be used in place of Equations (24), (29), (31), and (33). The results are summarized in Table 7.

Table 7. Recommended parameter range for Rosin-Rammler (a.k.a., Nukiyama-Tanasawa or generalized Gamma) size distributions assuming that *l* and *n* do not depend on the choice of the average fragment size.

Average Size	$n > 0$	$n < 0$
D_{avg}, M_{avg}	$l > 0$	$l < -m - 2$
$D_{M\ avg}$, $M_{M\ avg}$	$l > 0$	$l < -2m - 1$
D'_{avg}, M'_{avg}	$l > m - 1$	$l < -m - 2$
$D'_{M\ avg}$, $M'_{M\ avg}$	$l > 0$	$l < -m - 2$

4.5 Self-Similarity Condition 3

The third self-similarity condition requires that changing the average fragment size does not change the size distribution. In general, even when the size distribution stays the same, all the parameters change; an iterative technique is required to find the new parameters. However, suppose that the parameters *l* and *n* are invariant. Using Tables 2, 3, and 6, it can be shown that:

$$Q = \left(\frac{b_M}{b}\right)^{1/n} = \left(\frac{A_M}{A}\right)^{-1/(l+1)} \tag{35}$$

This, in turn, implies that Equations (23) and (28) are the same. Similarly, using Tables 2, 4, and 6, it can be shown that:

$$R = \left(\frac{b'}{b}\right)^{-1/n} = \left(\frac{A'}{A}\right)^{1/(l+1)} \tag{36}$$

This, in turn, implies that Equations (23) and (30) are the same. Finally, using Tables 2, 5, and 6, it can be shown that:

$$R_M = \left(\frac{b'_M}{b}\right)^{-1/n} = \left(\frac{A'_M}{A}\right)^{1/(l+1)} \tag{37}$$

This, in turn, implies that Equations (23) and (32) are the same.

5. Alternative Three-Parameter Form

5.1 Introduction

In Equation (23), replace D by M and D_{avg} by M_{avg} to obtain:

$$f(M) = \frac{1}{AM_{avg}}\left(\frac{M}{M_{avg}}\right)^{l} \exp\left[-b\left(\frac{M}{M_{avg}}\right)^{n}\right] \tag{38}$$

5.2 Transformation Condition

Using the expressions from Section 2, Equation (38) can be recast into eight equivalent forms; see Table 8.

Table 8a. Alternative three-parameter size distributions expressed in terms of *F* and *f*.

$F(D) = \frac{\Gamma\left[\frac{l+1}{n}, d\left(\frac{D}{D_{avg}}\right)^{mn}\right]}{\Gamma\left(\frac{l+1}{n}\right)}$	$F(M) = \frac{\Gamma\left[\frac{l+1}{n}, b\left(\frac{M}{M_{avg}}\right)^{n}\right]}{\Gamma\left(\frac{l+1}{n}\right)}$
$f(D) = \frac{m}{ASD_{avg}}\left(\frac{D}{SD_{avg}}\right)^{m(l+1)-1} \exp\left[-d\left(\frac{D}{D_{avg}}\right)^{mn}\right]$	$f(M) = \frac{1}{AM_{avg}}\left(\frac{M}{M_{avg}}\right)^{l} \exp\left[-b\left(\frac{M}{M_{avg}}\right)^{n}\right]$

Table 8b. Alternative three-parameter size distributions expressed in terms of F_M and f_M.

$F_M(D) = \frac{\Gamma\left[\frac{l+2}{n}, d\left(\frac{D}{D_{avg}}\right)^{mn}\right]}{\Gamma\left(\frac{l+2}{n}\right)}$	$F_M(M) = \frac{\Gamma\left[\frac{l+2}{n}, b\left(\frac{M}{M_{avg}}\right)^{n}\right]}{\Gamma\left(\frac{l+2}{n}\right)}$
$f_M(D) = \frac{m}{BSD_{avg}}\left(\frac{D}{SD_{avg}}\right)^{m(l+2)-1} \exp\left[-d\left(\frac{D}{D_{avg}}\right)^{mn}\right]$	$f_M(M) = \frac{1}{BM_{avg}}\left(\frac{M}{M_{avg}}\right)^{l+1} \exp\left[-b\left(\frac{M}{M_{avg}}\right)^{n}\right]$

The expressions in Table 8 are valid if $s > 0$ and $x > 0$ in $\Gamma(s, x)$. The first condition is true if:

$$\frac{l+1}{n} > 0 \quad \text{and} \quad \frac{l+2}{n} > 0$$

Equivalently:

$$l > -1 \text{ and } n > 0 \tag{39a}$$

or:

$$l < -2 \text{ and } n < 0 \tag{39b}$$

The second condition is true if:

$$b > 0;\ d > 0 \tag{40}$$

Equation (39) changes if the choice of the average fragment size changes; see Section 5.3. In addition, Equation (39) allows physical size distributions with infinite R or R_M; see Section 5.4 for more restrictive conditions that avoid this.

5.3 Self-Similarity Condition 1

The first self-similarity condition requires that Equation (37) obtain the correct D_{avg} and M_{avg}. Table 9 shows the parameter settings required to ensure the first self-similarity condition.

Table 9. Relationships among parameters for alternative three-parameter size distributions which ensure the correct D_{avg} and M_{avg}.

b	$W_0 = W_1$	$\Gamma\left(\frac{l+2}{n}\right)^n \Big/ \Gamma\left(\frac{l+1}{n}\right)^n$
d	$bS^{=mn}$	$\Gamma\left(\frac{l+1/m+1}{n}\right)^{mn} \Big/ \Gamma\left(\frac{l+1}{n}\right)^{mn}$
S	$\frac{W_0}{W_{1/m}}$	$b^{\frac{1}{mn}}\,\Gamma\left(\frac{l+1}{n}\right) \Big/ \Gamma\left(\frac{l+1/m+1}{n}\right)$
A	W_0	$\frac{1}{\lvert n\rvert}\, b^{-\frac{i+1}{n}}\,\Gamma\left(\frac{l+1}{n}\right)$
B	W_1	$\frac{1}{\lvert n\rvert}\, b^{-\frac{i+2}{n}}\,\Gamma\left(\frac{l+2}{n}\right)$

Notice that, as expected, all of the expressions in Table 9 are valid if Equation (39) is true.

Suppose $M_{M\,avg}$ is used as the reference mass instead of M_{avg}. Then Equation (38) becomes:

$$f(M) = \frac{1}{A_M M_{M\,avg}} \left(\frac{M}{M_{M\,avg}}\right)^{l_M} \exp\left[-b_M \left(\frac{M}{M_{M\,avg}}\right)^{n_M}\right] \tag{41}$$

In this case, Table 10 replaces Table 9.

Table 10. Relationships among parameters for alternative three-parameter size distributions which ensure the correct $D_{M\,avg}$ and $M_{M\,avg}$.

b_M	$W_1 = W_2$	$\Gamma\left(\frac{l_M+3}{n_M}\right)^{n_M} \Big/ \Gamma\left(\frac{l_M+2}{n_M}\right)^{n_M}$
d_M	$b_M S_M^{-mn_M}$	$\Gamma\left(\frac{l_M+1/m+2}{n_M}\right)^{mn_M} \Big/ \Gamma\left(\frac{l_M+2}{n_M}\right)^{mn_M}$
S_M	$\frac{W_1}{W_{1/m+1}}$	$b_M^{\frac{1}{mn_M}}\,\Gamma\left(\frac{l_M+2}{n_M}\right) \Big/ \Gamma\left(\frac{l_M+1/m+2}{n_M}\right)$
A_M	W_0	$\frac{1}{\lvert n_M\rvert}\, b_M^{-\frac{l_M+1}{n_M}}\,\Gamma\left(\frac{l_M+1}{n_M}\right)$
B_M	W_1	$\frac{1}{\lvert n_M\rvert}\, b_M^{-\frac{l_M+2}{n_M}}\,\Gamma\left(\frac{l_M+2}{n_M}\right)$

Notice that all the expressions in Table 10 are valid if:

$$\frac{l_M+1}{n_M} > 0 \quad \text{and} \quad \frac{l_M+3}{n_M} > 0$$

Equivalently:

$$l_M > -1 \text{ and } n_M > 0 \tag{42a}$$

or:

$$l_M < -3 \text{ and } n_M < 0 \tag{42b}$$

In other words, Equation (42) replaces Equation (39) when $M_{M\,avg}$ replaces M_{avg}. Similar expressions may be obtained when M'_{avg} or $M'_{M\,avg}$ replaces M_{avg}.

5.4 Self-Similarity Condition 2

The second self-similarity condition requires that changing the average fragment size does not change certain key parameters. For example, suppose the invariant parameters are *l* and *n*. In other words:

$$l = l_M$$
$$n = n_M$$

Then Table 11 gives expressions for ratios of averages for Equations (38) and (41).

Table 11. Ratios of averages for alternative three-parameter size distributions assuming *l* and *n* are invariant.

Q	$\dfrac{W_0}{W_1}\dfrac{W_{1/m+1}}{W_{1/m}}$	$\dfrac{\Gamma\left(\frac{l+1}{n}\right)\Gamma\left(\frac{l+1/m+2}{n}\right)}{\Gamma\left(\frac{l+2}{n}\right)\Gamma\left(\frac{l+1/m+1}{n}\right)}$
R	$\dfrac{W_{-1/m}W_{1/m}}{W_0^2}$	$\dfrac{\Gamma\left(\frac{l+1/m+1}{n}\right)\Gamma\left(\frac{l-1/m+1}{n}\right)}{\Gamma\left(\frac{l+1}{n}\right)^2}$
R_M	$\dfrac{W_{-1/m+1}W_{1/m+1}}{W_1^2}$	$\dfrac{\Gamma\left(\frac{l+1/m+2}{n}\right)\Gamma\left(\frac{l-1/m+2}{n}\right)}{\Gamma\left(\frac{l+2}{n}\right)^2}$

Notice that the expressions in Table 11 are valid if:

$$\frac{l-1/m+1}{n}>0 \quad \text{and} \quad \frac{l+1/m+2}{n}>0$$

Equivalently:

$$l>1/m-1 \text{ and } n>0 \tag{43a}$$

or:

$$l<-1/m-2 \text{ and } n<0 \tag{43b}$$

Where it is more restrictive, it is recommendation that Equation (43) be used in place of Equations (39) or (42).

5.5 Self-Similarity Condition 3

The third self-similarity condition requires that changing the average fragment size does not change the size distribution. In general, even when the size distribution stays the same, all the parameters change; an iterative technique is required to find the new parameters. However, suppose that the parameters *l* and *n* are invariant. Then using Tables 9, 10, and 11, it can be shown that Equations (38) and (41) are the same. A similar approach applies for other average sizes.

6. Two-Parameter Form: Gamma Size Distributions

6.1 Introduction

Dumouchel (2009) and Dumouchel et. al. (2012) note that the general form given in Section 4 has "a problem of parameter stability that manifests by drastic variations of their values for reasonable changes in the initial conditions." This and the following sections consider possible two-parameter subsets with more stable parameters. Consider the following size distributions:

Type I:
$$f_M(D) = \frac{1}{BD_{avg}}\left(\frac{D}{D_{avg}}\right)^{b-1} \exp\left[-b\left(\frac{D}{D_{avg}}\right)^{n}\right] \tag{44}$$

Type II:
$$f(D) = \frac{1}{AD_{avg}}\left(\frac{D}{D_{avg}}\right)^{b-1} \exp\left[-b\left(\frac{D}{D_{avg}}\right)^{n}\right] \tag{45}$$

Type III:
$$f_M(M) = \frac{1}{BM_{avg}}\left(\frac{M}{M_{avg}}\right)^{b-1} \exp\left[-b\left(\frac{M}{M_{avg}}\right)^{n}\right] \tag{46}$$

Type IV:
$$f(M) = \frac{1}{AM_{avg}}\left(\frac{M}{M_{avg}}\right)^{b-1} \exp\left[-b\left(\frac{M}{M_{avg}}\right)^{n}\right] \tag{47}$$

Equations (44) to (47) will be referred to as *Gamma size distributions*. Type I and II Gamma size distributions are examples of Rosin-Rammler (a.k.a Nukiyama-Tanasawa or generalized Gamma) size distributions as discussed in Section 4 with:

Type I: $l = b - m - 1$ (i.e., $k = b - 1$) (48a)
Type II: $l = b - 1$ (48b)

Similarly, Type III and IV Gamma size distributions are examples of the alternative three-parameter size distributions as discussed in Section 5 with:

Type III: $l = b - 2$ (i.e., $k = b - 1$) (49a)
Type IV: $l = b - 1$ (49b)

Until recently, Gamma size distributions were rarely used for fragmentation or its reverse, coagulation. In their first known appearance, Melzak (1953) showed that, if the initial size distribution is a Type IV Gamma size distribution, then the Smoluchowsk coagulation equation has an analytical solution; see also Friedlander & Wang (1966), Scott (1968), and Lindblad (2005, 2007).

As another example, consider fragmentation of unitary solid or liquid bodies. Based on computational results for Veronoi tesselations, Kiang (1966) suggested using Type IV Gamma size distributions. Since then, researchers have actively debated whether two-parameter Gamma size distributions can be used, as Kiang conjectured, or whether full three-parameter generalized Gamma distributions must be used instead, e.g., Hinde & Miles (1980), Tanemura (1988, 2003), Okabe et. al. (2000), Ferenc & Néda (2007).

More recently, Villermaux et. al. (2004) used Type II Gamma size distributions to describe "spray formed when a fast gas stream blows over a liquid volume." As a theoretical justification, they showed that Gamma size distributions are analytical solutions of Smoluchowski-type coagulation equations, reversing the role of input and output relative to Melzak (1953), Friedlander & Wang (1966), Scott (1968), and Lindblad (2005, 2007). To extend this observation from coagulation to fragmentation, they argued that a common liquid "fragmentation mechanism …, somewhat surprisingly, consists of a coalescence process."

While this may be true, there are at least two other possible explanations. First, based on a literature review, Vázquez & Gañán-Calvo (2010) suggested a simple "analogy between the equations of coalescence and fragmentation." Alternatively, in certain cases, coagulation and fragmentation may be inverse processes. Bertoin (2006) notes that "in general, time-reversal does not transform a fragmentation process into a coalescent process, and vice-versa" but that in selected cases there may be "remarkable duality between coagulation and fragmentation;" see also Dong et. al. (2006). The properties of Gamma size distributions, such as their close relationship to Poisson-Dirichlet distributions as described in Feller (1971), may qualify them for such duality. If so, the fact that Gamma size distributions are a natural initial state for coagulation, as noted above, means that they may also be a natural final state for fragmentation.

For other examples of Type II Gamma size distributions see, e.g., Marmottant & Villermaux (2004a, 2004b), Bremond & Villermaux (2006), Bremond et. al. (2007), Villermaux (2007), Eggers & Villermaux (2008), Villermaux & Bossa (2009, 2011), and Lhuissier & Villermaux (2013). In general, these papers treat the given Gamma size distributions as universals. In particular, Villermaux & Eggers (2008) note that "universality means that breakup is difficult to control, since its characteristics are independent of initial conditions."

6.2 Transformation Conditions

Using the expressions from Section 2, Equations (44) to (47) can each be recast into eight equivalent forms. As an example, Table 12 shows eight different forms for Type II Gamma size distributions. Table 12 is obtained by substituting $l = b - 1$ and $n = 1$ in Table 1.

Table 12a. Type II Gamma distributions for expressed in terms of *F* and *f*.

$F(D)=\dfrac{\Gamma\left[b, bD/D_{avg}\right]}{\Gamma(b)}$	$F(M)=\dfrac{\Gamma\left[b, d\left(M/M_{avg}\right)^{1/m}\right]}{\Gamma(b)}$
$f(D)=\dfrac{1}{AD_{avg}}\left(\dfrac{D}{D_{avg}}\right)^{b-1}\exp\left[-b\dfrac{D}{D_{avg}}\right]$	$f(M)=\dfrac{S}{mAM_{avg}}\left(\dfrac{SM}{M_{avg}}\right)^{b/m-1}\exp\left[-d\left(\dfrac{M}{M_{avg}}\right)^{1/m}\right]$

Table 12b. Type II Gamma distributions expressed in terms of F_M and f_M.

$F_M(D)=\dfrac{\Gamma\left[b+m, bD/D_{avg}\right]}{\Gamma(b+m)}$	$F_M(M)=\dfrac{\Gamma\left[b+m, d\left(M/M_{avg}\right)^{1/m}\right]}{\Gamma(b+m)}$
$f_M(D)=\dfrac{1}{BD_{avg}}\left(\dfrac{D}{D_{avg}}\right)^{b+m-1}\exp\left[-b\dfrac{D}{D_{avg}}\right]$	$f_M(M)=\dfrac{S}{mBM_{avg}}\left(\dfrac{SM}{M_{avg}}\right)^{b/m}\exp\left[-d\left(\dfrac{M}{M_{avg}}\right)^{1/m}\right]$

By Equations (24a) and (48), the transformation condition for Type I and II Gamma size distributions are as follows:

Type I: $b > m$ (50a)
Type II: $b > 0$ (50b)

By Equations (24b), (25), and (48), $b < 0$ and $n < 0$ are specifically excluded.

By Equations (39a) and (49), the transformation condition for Type III and IV Gamma size distribution are as follows:

Type III: $b > 1$ (51a)
Type IV: $b > 0$ (51b)

By Equations (38b), (39), and (49), $b < 0$ and $n < 0$ are, again, specifically excluded.

6.3 Self-Similarity Condition 1

As an example, let $l = b - m - 1$ in Table 2. Then for Type I Gamma size distributions, the correct D_{avg} is obtained if:

$$b = \Gamma\left(\frac{b-m+1}{n}\right)^n \Big/ \Gamma\left(\frac{b-m}{n}\right)^n \tag{52a}$$

Unfortunately, this expression lacks an analytical solution for *n;* solutions must be found iteratively. As another example, let $l = b - 1$ in Table 2. Then for Type II Gamma size distributions, the correct D_{avg} is obtained if:

$$b = \Gamma\left(\frac{b+1}{n}\right)^n \Big/ \Gamma\left(\frac{b}{n}\right)^n \tag{52b}$$

Fortunately, this has a very simple analytical solution. In particular, the relation $\Gamma(b+1) = b\Gamma(b)$ implies $n = 1$.

Similar results apply for Type III and IV Gamma size distributions; see Table 13 for an overall summary. Table 13 is derived by substituting Equations (48) and (49) into Tables 2 and 9.

Table 13. Parameters for Gamma size distributions which ensure the correct D_{avg} and M_{avg}.

	I	II	III	IV
n	$b = \dfrac{\Gamma\left(\frac{b-m+1}{n}\right)^n}{\Gamma\left(\frac{b-m}{n}\right)^n}$	1	$b = \dfrac{\Gamma\left(\frac{b}{n}\right)^n}{\Gamma\left(\frac{b-1}{n}\right)^n}$	1
d	$\dfrac{\Gamma\left(\frac{b}{n}\right)^{\frac{n}{m}}}{\Gamma\left(\frac{b-m}{n}\right)^{\frac{n}{m}}}$	$\dfrac{\Gamma(b+m)^{1/m}}{\Gamma(b)^{1/m}}$	$\dfrac{\Gamma\left(\frac{b+1/m-1}{n}\right)^{mn}}{\Gamma\left(\frac{b-1}{n}\right)^{mn}}$	$\dfrac{\Gamma(b+1/m)^m}{\Gamma(b)^m}$
S	$b^{-\frac{m}{n}} \dfrac{\Gamma\left(\frac{b}{n}\right)}{\Gamma\left(\frac{b-m}{n}\right)}$	$\dfrac{1}{b^m} \dfrac{\Gamma(b+m)}{\Gamma(b)}$	$b^{\frac{1}{mn}} \dfrac{\Gamma\left(\frac{b-1}{n}\right)}{\Gamma\left(\frac{b+1/m-1}{n}\right)}$	$\dfrac{b^{1/m}\Gamma(b)}{\Gamma(b+1/m)}$
A	$\dfrac{1}{n} b^{-\frac{b-m}{n}} \Gamma\left(\frac{b-m}{n}\right)$	$\dfrac{\Gamma(b)}{b^b}$	$\dfrac{1}{n} b^{-\frac{b-1}{n}} \Gamma\left(\frac{b-1}{n}\right)$	$\dfrac{\Gamma(b)}{b^b}$
B	$\dfrac{1}{n} b^{-\frac{b}{n}} \Gamma\left(\frac{b}{n}\right)$	$\dfrac{\Gamma(b+m)}{b^{b+m}}$	$\dfrac{1}{n} b^{-\frac{b}{n}} \Gamma\left(\frac{b}{n}\right)$	$\dfrac{\Gamma(b+1)}{b^{b+1}}$

Notice that, as expected, all of the expressions in Table 13 are valid if Equations (50) and (51) are true.

Suppose $D_{M\,avg}$ is used as the reference diameter instead of D_{avg} and, similarly, $M_{M\,avg}$ is used as the reference mass instead of M_{avg}. Then Equations (44) to (47) become:

Type I: $$f_M(D) = \frac{1}{B_M D_{M\,avg}} \left(\frac{D}{D_{M\,avg}} \right)^{b_M - 1} \exp\left[-b_M \left(\frac{D}{D_{M\,avg}} \right)^{n_M} \right] \tag{53}$$

Type II: $$f(D) = \frac{1}{A_M D_{M\,avg}} \left(\frac{D}{D_{M\,avg}} \right)^{b_M - 1} \exp\left[-b_M \left(\frac{D}{D_{M\,avg}} \right)^{n_M} \right] \tag{54}$$

Type III: $$f_M(M) = \frac{1}{B_M M_{M\,avg}} \left(\frac{M}{M_{M\,avg}} \right)^{b_M - 1} \exp\left[-b_M \left(\frac{M}{M_{M\,avg}} \right)^{n_M} \right] \tag{55}$$

Type IV: $$f(M) = \frac{1}{A_M M_{M\,avg}} \left(\frac{M}{M_{M\,avg}} \right)^{b_M - 1} \exp\left[-b_M \left(\frac{M}{M_{M\,avg}} \right)^{n_M} \right] \tag{56}$$

As an example, let $l_M = b_M - m - 1$ in Table 3. Then for Type I Gamma size distributions, the correct $D_{M\,avg}$ is obtained if:

$$b_M = \Gamma\left(\frac{b_M + 1}{n_M} \right)^{n_M} \Big/ \Gamma\left(\frac{b_M}{n_M} \right)^{n_M} \tag{57a}$$

Fortunately, this has a very simple analytical solution. In particular, the relation $\Gamma(b_M + 1) = b_M \Gamma(b_M)$ implies $n_M = 1$.

As another example, let $l_M = b_M - 1$ in Table 3. Then for Type II Gamma size distributions, the correct $D_{M\,avg}$ is obtained if:

$$b_M = \Gamma\left(\frac{b_M + m + 1}{n_M} \right)^{n_M} \Big/ \Gamma\left(\frac{b_M + m}{n_M} \right)^{n_M} \tag{57b}$$

Unfortunately, this expression lacks an analytical solution for n_M; solutions must be found iteratively. Similar results apply for Type III and IV Gamma size distributions; see Table 14 for an overall summary.

Table 14. Parameters for Gamma size distributions which ensure the correct $D_{M\,avg}$ and $M_{M\,avg}$.

	I	II	III	IV
n_M	1	$b_M = \dfrac{\Gamma\left(\dfrac{b_M+m+1}{n_M}\right)^{n_M}}{\Gamma\left(\dfrac{b_M+m}{n_M}\right)^{n_M}}$	1	$b_M = \dfrac{\Gamma\left(\dfrac{b_M+2}{n_M}\right)^{n_M}}{\Gamma\left(\dfrac{b_M+1}{n_M}\right)^{n_M}}$
d_M	$\dfrac{\Gamma(b_M+m)^{1/m}}{\Gamma(b_M)^{1/m}}$	$\dfrac{\Gamma\left(\dfrac{b_M+2m}{n_M}\right)^{\frac{n_M}{m}}}{\Gamma\left(\dfrac{b_M+m}{n_M}\right)^{\frac{n_M}{m}}}$	$\dfrac{\Gamma(b_M+1/m)^m}{\Gamma(b_M)^m}$	$\dfrac{\Gamma\left(\dfrac{b_M+1/m+1}{n_M}\right)^{mn_M}}{\Gamma\left(\dfrac{b_M+1}{n_M}\right)^{mn_M}}$
S_M	$\dfrac{1}{b_M^m}\dfrac{\Gamma(b_M+m)}{\Gamma(b_M)}$	$\dfrac{\Gamma\left(\dfrac{b_M+2m}{n_M}\right)}{b_M^{\frac{m}{n_M}}\Gamma\left(\dfrac{b_M+m}{n_M}\right)}$	$\dfrac{b_M^{1/m}\Gamma(b_M)}{\Gamma(b_M+1/m)}$	$b_M^{\frac{1}{mn_M}}\dfrac{\Gamma\left(\dfrac{b_M+1}{n_M}\right)}{\Gamma\left(\dfrac{l+1/m+1}{n_M}\right)}$
A_M	$\dfrac{\Gamma(b_M-m)}{b^{b_M-m}}$	$\dfrac{1}{n_M}b_M^{-\frac{b_M}{n_M}}\Gamma\left(\dfrac{b_M}{n_M}\right)$	$\dfrac{\Gamma(b_M-1)}{b_M^{b_M-1}}$	$\dfrac{1}{n_M}b_M^{-\frac{b_M}{n_M}}\Gamma\left(\dfrac{b_M}{n_M}\right)$
B_M	$\dfrac{\Gamma(b_M)}{b_M^b}$	$\dfrac{1}{n_M}b_M^{-\frac{b_M+1}{n_M}}\Gamma\left(\dfrac{b_M+1}{n_M}\right)$	$\dfrac{\Gamma(b_M)}{b_M^{b_M}}$	$\dfrac{1}{n_M}b_M^{-\frac{b_M+1}{n_M}}\Gamma\left(\dfrac{b_M+1}{n_M}\right)$

Similar expressions may be obtained for M'_{avg}, D'_{avg} and $M'_{M\,avg}$, $D'_{M\,avg}$.

6.4 Self-Similarity Condition 2

The second self-similarity condition requires that changing the average fragment size does not change key parameters. Sections 4.4 and 5.4 assumed that *l* and *n* are invariant while *b* is non-invariant. However, for Gamma distributions, *l* is a function of *b*. Since *b* is non-invariant, *l* must also be non-invariant. Thus, in fact, Gamma size distributions have no invariant parameters.

To distinguish between the different parameters sets, A, B, C, and D may be used to refer to normalization by $D_{M\,avg}$, D_{avg}, $D'_{M\,avg}$, and D'_{avg}, respectively. For example, Table 14 gives expressions for ratios of fragment averages for Type IA, IIB, IIIA, and IVB Gamma size distributions.

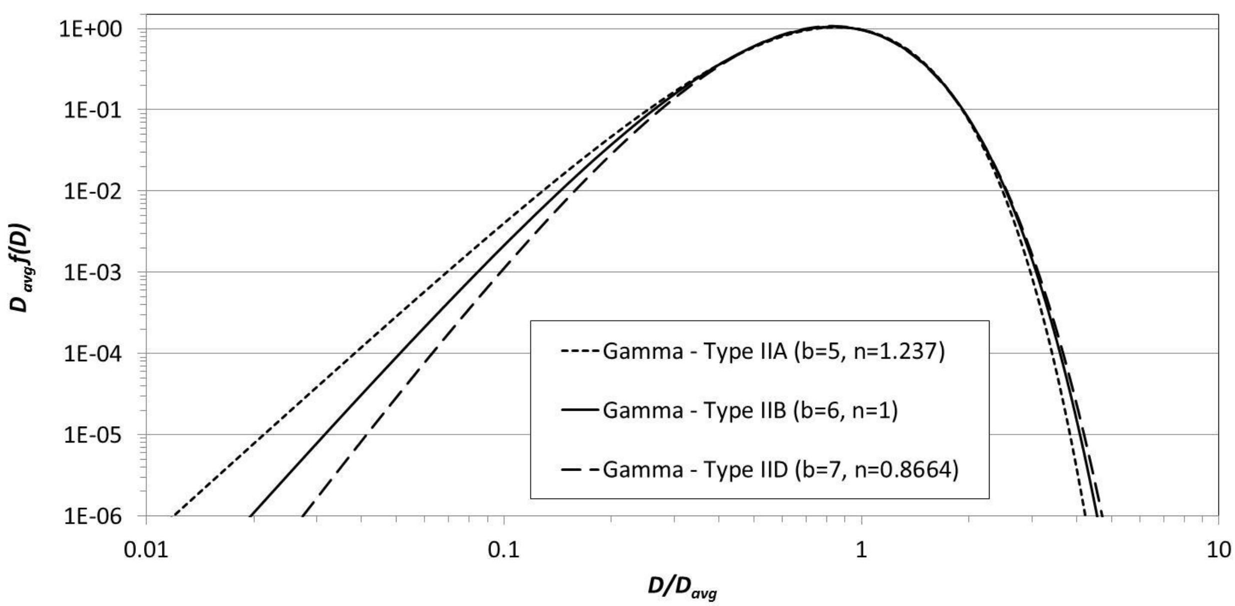
Table 15. Ratios of averages for selected Gamma size distributions.

	IA ($D_{M\,avg}$)	IIB (D_{avg})	IIIA ($D_{M\,avg}$)	IVB (D_{avg})
Q	$\frac{b_M}{b_M - m}$	$\frac{b+m}{b}$	$\frac{b_M - 1 + 1/m}{b_M - 1}$	$\frac{b + 1/m}{b}$
R	$\frac{b_M - m}{b_M - m - 1}$	$\frac{b}{b-1}$	$\frac{\Gamma\left(b_M - 1 + \frac{1}{m}\right)\Gamma\left(b_M - 1 - \frac{1}{m}\right)}{\Gamma(b_M - 1)^2}$	$\frac{\Gamma\left(b + \frac{1}{m}\right)\Gamma\left(b - \frac{1}{m}\right)}{\Gamma(b)^2}$
R_M	$\frac{b_M}{b_M - 1}$	$\frac{b+m}{b+m-1}$	$\frac{\Gamma\left(b_M + \frac{1}{m}\right)\Gamma\left(b_M - \frac{1}{m}\right)}{\Gamma(b_M)^2}$	$\frac{\Gamma\left(b + 1 + \frac{1}{m}\right)\Gamma\left(b + 1 - \frac{1}{m}\right)}{\Gamma(b+1)^2}$

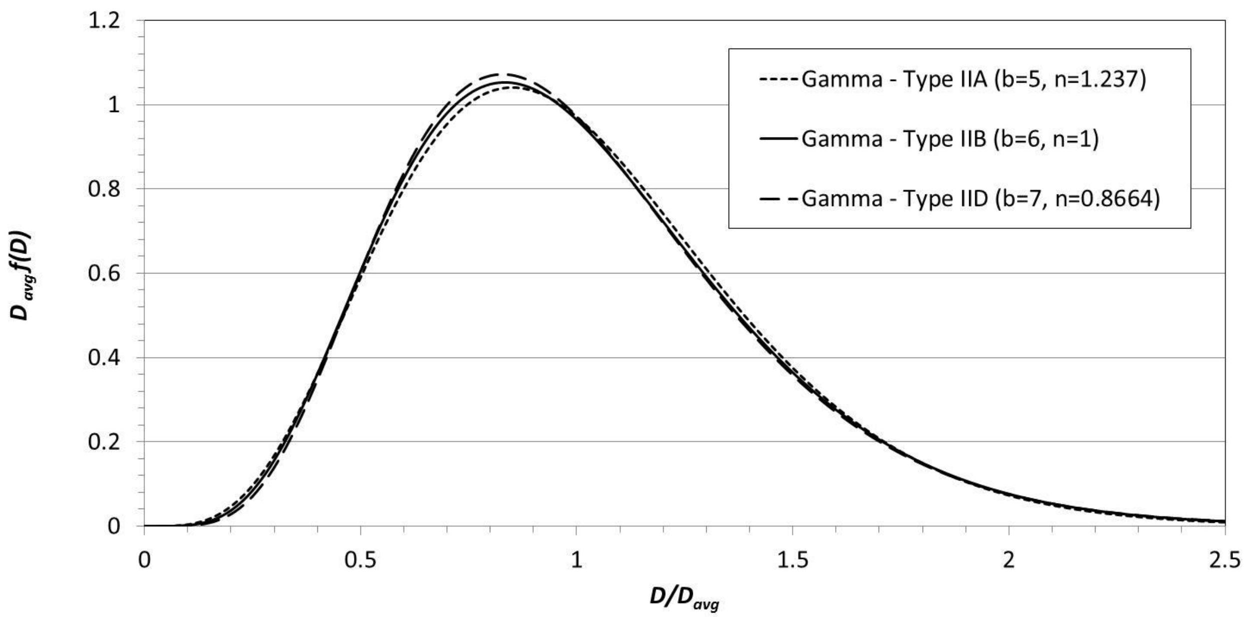
6.5 Self-Similarity Condition 3

As already noted, the parameters in the Gamma size distribution change when the normalization changes. However, the Gamma size distribution itself may remain approximately the same. In fact, Type IIA, IIB, and IID Gamma size distributions are nearly identical except for the smallest fragments.

Figure 1 shows an example where the Type IIA curve is plotted as $Q^{-1}\ f(Q^{-1}D)$, the Type IIB curve is plotted as $f(D)$, and the Type IID curve is plotted as $Rf(RD)$.



(a.) Log-log plane



(b.) Linear-linear plane

Figure 1. An example showing that Gamma size distributions may be approximately the same regardless of normalization. In particular, the Type IIA is normalized by (and ensures the correct value for) the mass mean diameter, the Type IIB is normalized by (and ensures the correct value for) the count mean diameter, and the Type IID is normalized by (and ensures the correct value for) the average defined by Equation (12).

7. Two-Parameter Form: Weibull Size Distributions

7.1 Introduction

Consider the following size distributions:

Type I: $$f_M(D) = \frac{1}{BD_{avg}}\left(\frac{D}{D_{avg}}\right)^{n-1}\exp\left[-b\left(\frac{D}{D_{avg}}\right)^{n}\right] \quad (58)$$

Type II: $$f(D) = \frac{1}{AD_{avg}}\left(\frac{D}{D_{avg}}\right)^{n-1}\exp\left[-b\left(\frac{D}{D_{avg}}\right)^{n}\right] \quad (59)$$

Type III: $$f_M(M) = \frac{1}{BM_{avg}}\left(\frac{M}{M_{avg}}\right)^{n-1}\exp\left[-b\left(\frac{M}{M_{avg}}\right)^{n}\right] \quad (60)$$

Type IV: $$f(M) = \frac{1}{AM_{avg}}\left(\frac{M}{M_{avg}}\right)^{n-1}\exp\left[-b\left(\frac{M}{M_{avg}}\right)^{n}\right] \quad (61)$$

Equations (58) to (61) will be referred to as *Weibull size distributions* after Weibull (1939a,b). For a modern treatment of Weibull distributions, see, e.g., Rinne (2008). Type I and II Weibull size distributions are examples of Rosin-Rammler (a.k.a. Nukiyama-Tanasawa or generalized Gamma) size distributions as discussed in Section 4 with:

Type I: $l = n - m - 1$ (i.e., $k = n - 1$) (62a)
Type II: $l = n - 1$ (62b)

Notice that Equation (62) is the same as Equation (48), except that *n* replaces *b*. Similarly, Type III and IV Weibull size distributions are examples of alternative three-parameter size distributions as discussed in Section 5 with:

Type III: $l = n - 2$ (i.e., $k = n - 1$) (63a)
Type IV: $l = n - 1$ (63b)

Notice that Equation (63) is the same as Equation (49), except that *n* replaces *b*.

It can be shown that Type I and III Weibull distributions are the same, provided that *n* in the Type III equals *n*/*m* in the Type I (and that the other parameters are adjusted accordingly). Similarly, Type II and IV Weibull distributions are the same, provided that *n* in the Type IV equals *n*/*m* in the Type II (and that the other parameters are adjusted accordingly).

Type I Weibull distributions are commonly known as *Rosin-Rammler-Sperling-Bennett (RRSB) distributions* after Rosin et. al. (1933) and Bennett (1936). Type II Weibull distributions are sometimes known as *generalized Mott-Linfoot distributions* after Mott & Linfoot (1943), e.g., Mock & Holt (1983), Grady et. al. (2001), Grady (2006), Arnold & Rottenkolber (2008).

As an example, the meteorology community commonly uses the Marshall-Palmer (1948) law for raindrop size distributions. The Marshall-Palmer law is a universal Type II Weibull size distribution with $n = 1$ and $m = 3$. Villermaux & Bossa (2009) showed that the Marshall-Palmer law can be "understood from the fragmentation products of non-interacting, isolated drops." As another example, the weapons effects community commonly uses the Mott-Linfoot (1943) distributions to describe metal casing fragments; see also, e.g., Grady (2006). The Mott-Linfoot distributions are universal Type II Weibull distributions with n/m equal to 1/2 or 1/3.

7.2 Transformation Condition

Using the expressions from Section 2, Equations (58) to (61) can each be recast into eight equivalent forms. As an example, Table 16 shows eight different forms for Type II Weibull size distributions. Table 16 is obtained by substituting $l = n-1$ in Table 1. Notice that the expression $\Gamma(1,x) = \exp(-x)$ has been used.

Table 16a. Type II Weibull size distributions expressed in terms of *F* and *f*.

$F(D) = \exp[-b(D/D_{avg})^n]$	$F(M) = \exp[-d\,(M/M_{avg})^{n/m}]$
$f(D) = \dfrac{1}{AD_{avg}}\left(\dfrac{D}{D_{avg}}\right)^{n-1}\exp\left[-b\left(\dfrac{D}{D_{avg}}\right)^n\right]$	$f(M) = \dfrac{S}{mAM_{avg}}\left(\dfrac{SM}{M_{avg}}\right)^{n/m-1}\exp\left[-d\left(\dfrac{M}{M_{avg}}\right)^{n/m}\right]$

Table 16b. Type II Weibull size distributions expressed in terms of F_M and f_M.

$F_M(D) = \dfrac{\Gamma\left[1+m/n,\, b\left(D/D_{avg}\right)^n\right]}{\Gamma(1+m/n)}$	$F_M(M) = \dfrac{\Gamma\left[1+m/n,\, d\left(M/M_{avg}\right)^{n/m}\right]}{\Gamma(1+m/n)}$
$f_M(D) = \dfrac{1}{BD_{avg}}\left(\dfrac{D}{D_{avg}}\right)^{n+m-1}\exp\left[-b\left(\dfrac{D}{D_{avg}}\right)^n\right]$	$f_M(M) = \dfrac{S}{mBM_{avg}}\left(\dfrac{SM}{M_{avg}}\right)^{n/m}\exp\left[-d\left(\dfrac{M}{M_{avg}}\right)^{n/m}\right]$

By Equations (24) and (62), the transformation conditions for Type I and II Weibull size distributions are as follows:

Type I: $n > m$ or $n < 0$ (64a)

Type II: $n > 0$ or $n < -m$ (64b)

By Equations (39) and (63), the transformation conditions for Type III and IV Weibull size distribution are as follows:

Type III: $n > 1$ or $n < 0$ (65a)

Type IV: $n > 0$ or $n < -1$ (65b)

When *n* is negative, which is rare, these are sometimes called *negative* or *inverse* or *reverse Weibull distributions*, e.g., Rinne (2008), Lai (2014). Negative Weibull distributions have mainly been used for reliability engineering rather than fragmentation.

7.3 Self-Similarity Condition 1

Table 17 shows the parameters which ensure that Equations (58) to (61) obtain the correct D_{avg} and M_{avg}. Table 17 is derived by substituting Equations (62) and (63) into Tables 2 and 9.

Table 17. Parameters for Weibull distributions which ensure the correct D_{avg} and M_{avg}.

	I	II	III	IV
b	$\dfrac{\Gamma\left(1-\frac{m-1}{n}\right)^{n}}{\Gamma\left(1-\frac{m}{n}\right)^{n}}$	$\Gamma\left(1+\frac{1}{n}\right)^{n}$	$\Gamma\left(1-\frac{1}{n}\right)^{-n}$	$\Gamma\left(1+\frac{1}{n}\right)^{n}$
d	$\Gamma\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}$	$\Gamma\left(1+\frac{m}{n}\right)^{\frac{n}{m}}$	$\dfrac{\Gamma\left(1-\frac{m-1}{mn}\right)^{mn}}{\Gamma\left(1-\frac{1}{n}\right)^{mn}}$	$\Gamma\left(1+\frac{1}{mn}\right)^{mn}$
S	$\dfrac{b^{-\frac{m}{n}}}{\Gamma\left(1-\frac{m}{n}\right)}$	$b^{-\frac{m}{n}}\,\Gamma\left(1+\frac{m}{n}\right)$	$b^{\frac{1}{mn}}\dfrac{\Gamma\left(1-\frac{1}{n}\right)}{\Gamma\left(1-\frac{m-1}{mn}\right)}$	$\dfrac{b^{\frac{1}{mn}}}{\Gamma\left(1+\frac{1}{mn}\right)}$
A	$\frac{1}{\lvert n\rvert}\,b^{-\left(1-\frac{m}{n}\right)}\,\Gamma\left(1-\frac{m}{n}\right)$	$\frac{1}{\lvert n\rvert b}$	$\frac{1}{\lvert n\rvert}\,b^{-\left(1-\frac{1}{n}\right)}\,\Gamma\left(1-\frac{1}{n}\right)$	$\frac{1}{\lvert n\rvert b}$
B	$\frac{1}{\lvert n\rvert b}$	$\frac{1}{\lvert n\rvert}\,b^{-\left(1+\frac{m}{n}\right)}\,\Gamma\left(1+\frac{m}{n}\right)$	$\frac{1}{\lvert n\rvert b}$	$\frac{1}{\lvert n\rvert}\,b^{-\left(1+\frac{1}{n}\right)}\,\Gamma\left(1+\frac{1}{n}\right)$

Suppose $D_{M\,avg}$ is used as the reference diameter instead of D_{avg} and, similarly, $M_{M\,avg}$ is used as the reference mass instead of M_{avg}. Assume that this change does not affect the key parameter *n*. Then Equations (58) to (61) become:

Type I: $$f_M(D) = \frac{1}{B_M D_{M\,avg}} \left(\frac{D}{D_{M\,avg}} \right)^{n-1} \exp\left[-b_M \left(\frac{D}{D_{M\,avg}} \right)^n \right] \tag{66}$$

Type II: $$f(D) = \frac{1}{A_M D_{M\,avg}} \left(\frac{D}{D_{M\,avg}} \right)^{n-1} \exp\left[-b_M \left(\frac{D}{D_{M\,avg}} \right)^n \right] \tag{67}$$

Type III: $$f_M(M) = \frac{1}{B_M M_{M\,avg}} \left(\frac{M}{M_{M\,avg}} \right)^{n-1} \exp\left[-b_M \left(\frac{M}{M_{M\,avg}} \right)^n \right] \tag{68}$$

Type IV: $$f(M) = \frac{1}{A_M M_{M\,avg}} \left(\frac{M}{M_{M\,avg}} \right)^{n-1} \exp\left[-b_M \left(\frac{M}{M_{M\,avg}} \right)^n \right] \tag{69}$$

Then Table 18 replaces Table 17.

Table 18. Parameters for Weibull distributions which ensure the correct $D_{M\,avg}$ and $M_{M\,avg}$.

	I	II	III	IV
b_M	$\Gamma\left(1+\frac{1}{n}\right)^n$	$\dfrac{\Gamma\left(1+\frac{m+1}{n}\right)^n}{\Gamma\left(1+\frac{m}{n}\right)^n}$	$\Gamma\left(1+\frac{1}{n}\right)^n$	$\dfrac{\Gamma\left(1+\frac{2}{n}\right)^n}{\Gamma\left(1+\frac{1}{n}\right)^n}$
d_M	$\Gamma\left(1+\frac{m}{n}\right)^{\frac{n}{m}}$	$\dfrac{\Gamma\left(1+\frac{2m}{n}\right)^{\frac{n}{m}}}{\Gamma\left(1+\frac{m}{n}\right)^{\frac{n}{m}}}$	$\Gamma\left(1+\frac{1}{nm}\right)^{mn}$	$\dfrac{\Gamma\left(1+\frac{m+1}{nm}\right)^{mn}}{\Gamma\left(1+\frac{1}{n}\right)^{mn}}$
S_M	$b_M^{-\frac{m}{n}}\ \Gamma\left(1+\frac{m}{n}\right)$	$b_M^{-\frac{m}{n}}\ \dfrac{\Gamma\left(1+\frac{2m}{n}\right)}{\Gamma\left(1+\frac{m}{n}\right)}$	$\dfrac{b_M^{\frac{1}{mn}}}{\Gamma\left(1+\frac{1}{nm}\right)}$	$b_M^{\frac{1}{mn}}\ \dfrac{\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(1+\frac{m+1}{nm}\right)}$
A_M	$\frac{1}{\lvert n\rvert}\, b_M^{-\left(1-\frac{m}{n}\right)}\ \Gamma\left(1-\frac{m}{n}\right)$	$\dfrac{1}{\lvert n\rvert b_M}$	$\frac{1}{\lvert n\rvert}\, b_M^{-\left(1-\frac{1}{n}\right)}\ \Gamma\left(1-\frac{1}{n}\right)$	$\dfrac{1}{\lvert n\rvert b_M}$
B_M	$\dfrac{1}{\lvert n\rvert b_M}$	$\frac{1}{\lvert n\rvert}\, b_M^{-\left(1+\frac{m}{n}\right)}\ \Gamma\left(1+\frac{m}{n}\right)$	$\dfrac{1}{\lvert n\rvert b_M}$	$\frac{1}{\lvert n\rvert}\, b_M^{-\left(1+\frac{1}{n}\right)}\ \Gamma\left(1+\frac{1}{n}\right)$

Similar expressions may be obtained for M'_{avg}, D'_{avg} and $M'_{M\,avg}$, $D'_{M\,avg}$.

7.4 Self-Similarity Condition 2

The second self-similarity condition requires that changing the average fragment size does not change key parameters. Sections 4.4 and 5.4 assumed that l and n were invariant. For Weibull distributions, l is a function of the invariant n. Thus both l and n are, in fact, invariant.

Table 19 gives expressions for ratios of fragment averages. Table 19a is obtained by substituting Equation (64) in Table 6. Similarly, Table 19b is obtained by substituting Equation (65) into Table 11.

Table 19a. Ratios of averages for Type I and II Weibull distributions.

	I	II
Q	$\dfrac{\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1-\frac{m}{n}\right)}{\Gamma\left(1-\frac{m-1}{n}\right)}$	$\dfrac{\Gamma\left(1+\frac{m+1}{n}\right)}{\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1+\frac{m}{n}\right)}$
R	$\dfrac{\Gamma\left(1-\frac{m+1}{n}\right)\Gamma\left(1-\frac{m-1}{n}\right)}{\Gamma\left(1-\frac{m}{n}\right)^2}$	$\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)$
R_M	$\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)$	$\dfrac{\Gamma\left(1+\frac{m+1}{n}\right)\Gamma\left(1+\frac{m-1}{n}\right)}{\Gamma\left(1+\frac{m}{n}\right)^2}$

Table 19b. Ratios of averages for Type III and IV Weibull distributions.

	III	IV
Q	$\dfrac{\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1+\frac{1}{mn}\right)}{\Gamma\left(1-\frac{m-1}{mn}\right)}$	$\dfrac{\Gamma\left(1+\frac{m+1}{mn}\right)}{\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1+\frac{1}{mn}\right)}$
R	$\dfrac{\Gamma\left(1-\frac{m-1}{mn}\right)\Gamma\left(1-\frac{m+1}{mn}\right)}{\Gamma\left(1-\frac{1}{n}\right)^2}$	$\Gamma\left(1+\frac{1}{mn}\right)\Gamma\left(1-\frac{1}{mn}\right)$
R_M	$\Gamma\left(1+\frac{1}{mn}\right)\Gamma\left(1-\frac{1}{mn}\right)$	$\dfrac{\Gamma\left(1+\frac{m-1}{mn}\right)\Gamma\left(1+\frac{m+1}{mn}\right)}{\Gamma\left(1+\frac{1}{n}\right)^2}$

7.4 Self-Similarity Condition 3

The proofs given in Section 4.5 and 5.5 apply. In other words, when n is fixed and the other parameters vary as in Table 18, Weibull distributions are the same regardless of the choice of average fragment size.

8. Modified Gamma and Weibull Distributions

As defined here, Type I and II *modified Gamma size distributions* are examples of Rosin-Rammler (a.k.a Nukiyama-Tanasawa or generalized Gamma) size distributions as discussed in Section 4 with:

Type I: $l = ib - m - 1$ (70a)
Type II: $l = ib - 1$ (70b)

where i is any integer greater than 1. Notice that Equation (70) is the same as Equation (48), except that ib replaces b. Similarly, Type III and IV modified Gamma size distributions are examples of the alternative three-parameter size distributions as discussed in Section 5 with:

Type III: $l = ib - 2$ (71a)
Type IV: $l = ib - 1$ (71b)

where i is any integer greater than 1. Notice that Equation (71) is the same as Equation (49), except that ib replaces b.

As defined here, Type I and II *modified Weibull size distributions* are examples of Rosin-Rammler (a.k.a Nukiyama-Tanasawa or generalized Gamma) size distributions as discussed in Section 4 with:

Type I: $l = in - m - 1$ (72a)
Type II: $l = in - 1$ (72b)

where i is any integer greater than 1. Notice that Equation (72) is the same as Equation (62), except that in replaces n. Similarly, Type III and IV modified Gamma size distributions are examples of the alternative three-parameter size distributions as discussed in Section 5 with:

Type III: $l = in - 2$ (73a)
Type IV: $l = in - 1$ (73b)

where i is any integer greater than 1. Notice that Notice that Equation (73) is the same as Equation (63), except that in replaces n.

For example, suppose $i = 2$. Then modified Weibull distributions can be written as follows:

$$\text{Type I:} \qquad F_M(D) = \left[1 + b\left(\frac{D}{D_{avg}}\right)^n\right] \exp\left[-b\left(\frac{D}{D_{avg}}\right)^n\right] \tag{74}$$

$$\text{Type II:} \qquad F(D) = \left[1 + b\left(\frac{D}{D_{avg}}\right)^n\right] \exp\left[-b\left(\frac{D}{D_{avg}}\right)^n\right] \tag{75}$$

$$\text{Type III:} \qquad F_M(M) = \left[1 + b\left(\frac{M}{M_{avg}}\right)^n\right] \exp\left[-b\left(\frac{M}{M_{avg}}\right)^n\right] \tag{76}$$

$$\text{Type IV:} \qquad F(M) = \left[1 + b\left(\frac{M}{M_{avg}}\right)^n\right] \exp\left[-b\left(\frac{M}{M_{avg}}\right)^n\right] \tag{77}$$

The expression $\Gamma(2, x) = (1 + x)\exp(-x)$ has been used.

Levy et. al. (2010) first suggested Type IV modified Weibull distributions. Daphalapurkar (2015) later suggested Type II modified Weibull distributions. Limited comparisons conducted to date indicate that modified Weibull and Gamma distributions are, for all practical purposes, identical to Weibull and Gamma distributions. They are included here for the sake of completeness.

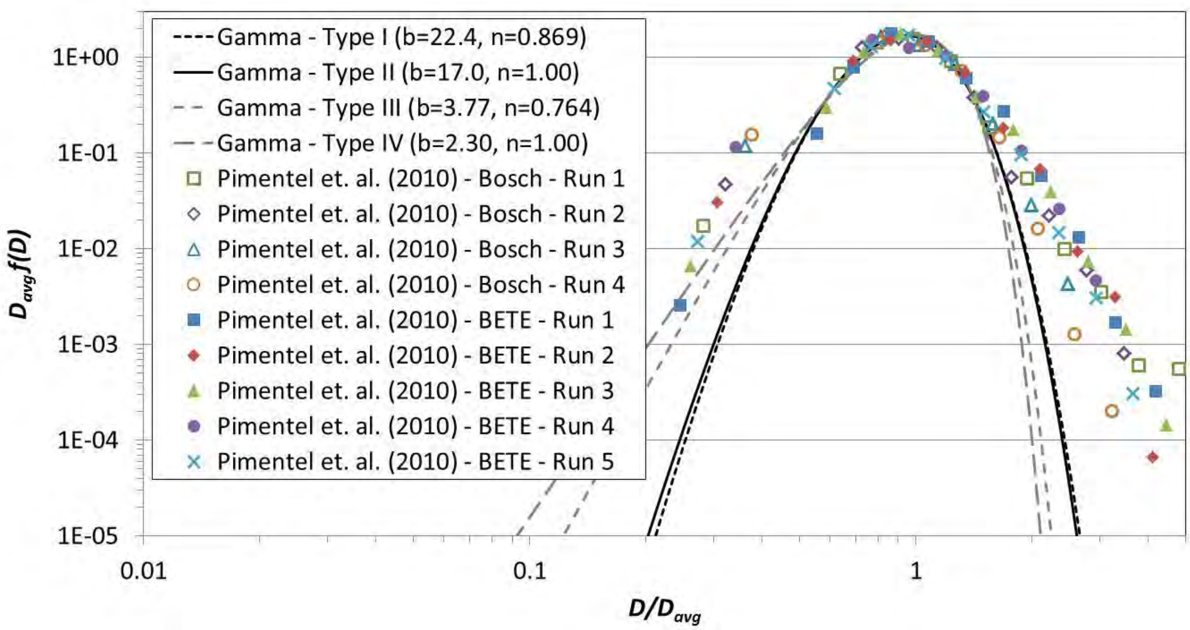
9. Comparisons to Test Data

This section compares Gamma and Weibull size distribution to liquid spray atomization test data. Small size spread data is taken from Pimentel et. al. (2010), medium size spread data is taken from Li & Tankin (1987) and Tishkoff (1979), and large size spread data is taken from Tuner & Moulton (1953).

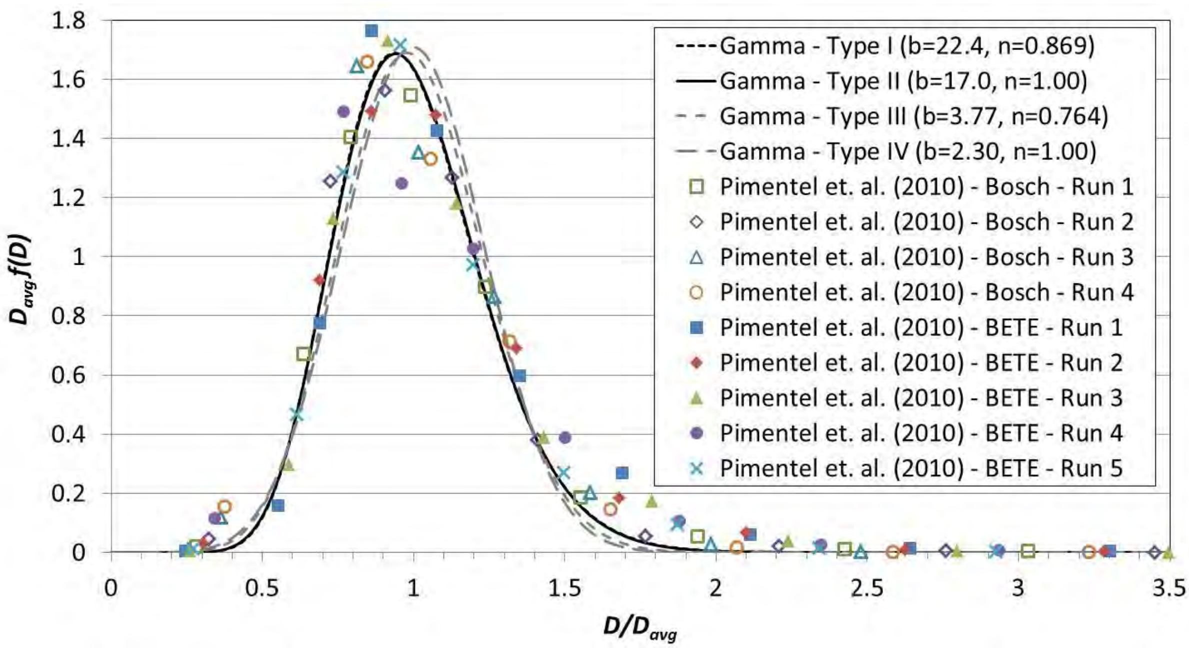
9.1 Gamma Size Distributions

Figures 2, 3, and 4 compare Gamma size distributions to small, medium, and large size spread data, respectively. The Type II Gamma size distributions were taken from Villermaux (2007) and Marmottant & Villermaux (2004a, b). The Type I, III, and IV Gamma size distributions were obtained by matching R to the Type II Gamma size distributions.

As seen in these examples, Gamma size distributions typically match test data well, albeit the agreement sometimes deteriorates at the extremes. With the parameter choices made here, Type I and II Gamma distributions generally outperform Type III and IV Gamma distributions.

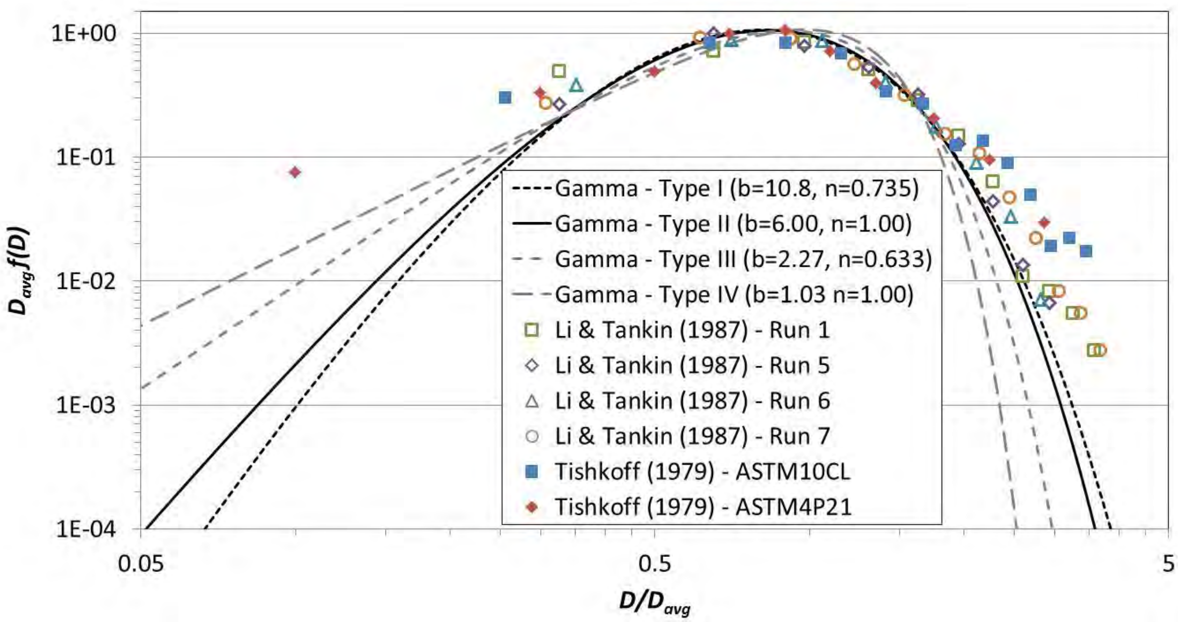


(a.) Log-log plane

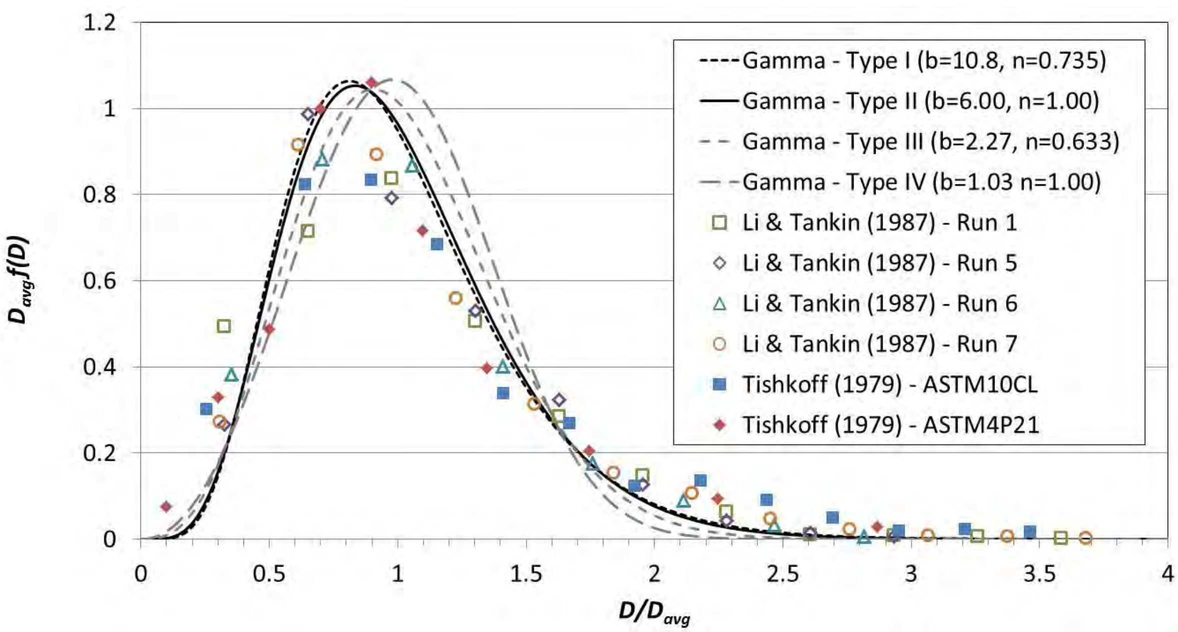


(b.) Linear-Linear Plane

Figure 2. Gamma size distributions vs. small size spread test data for spray atomization taken from Pimentel et. al. (2010). The Type II parameter b=17 is taken from Villermaux (2007). The Type I, III, and IV parameters are chosen so that all size distributions have R=1.0625. In all cases, m=3.

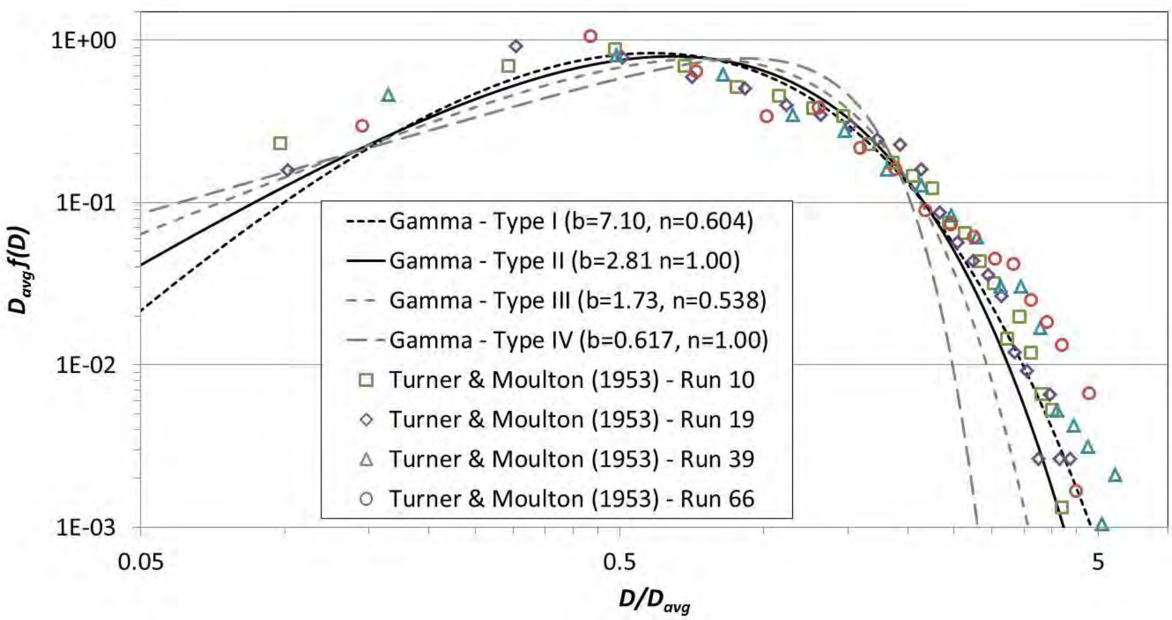


(a.) Log-log plane

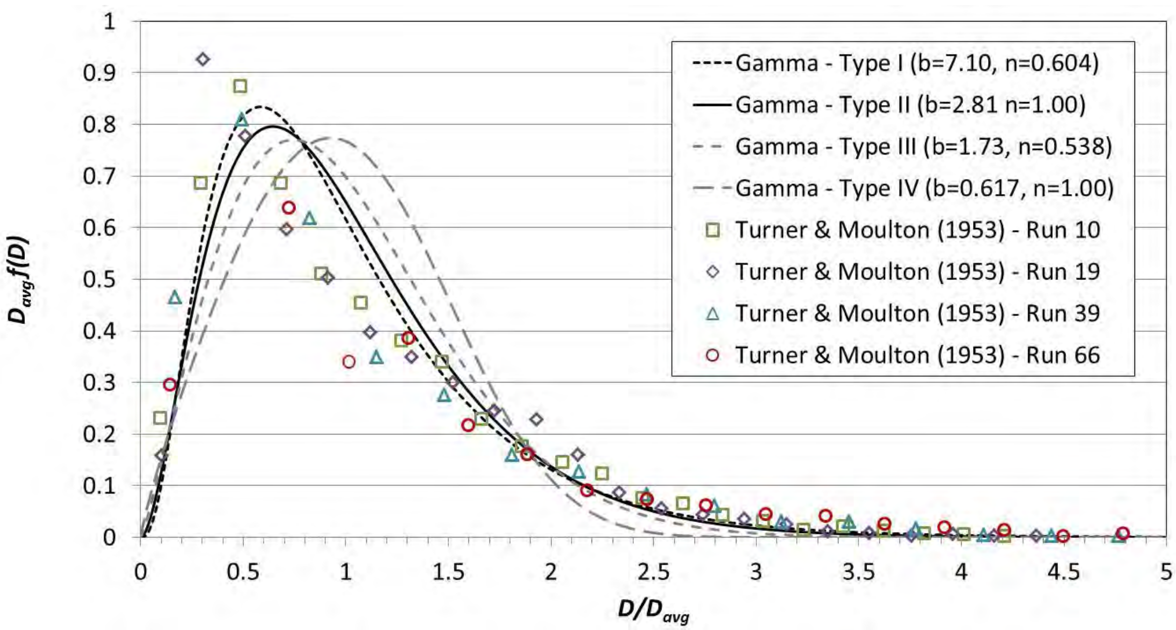


(b.) Linear-linear plane

Figure 3. Gamma size distributions vs. medium size spread test data for spray atomization taken from Li & Tankin (1987) and Tishkoff (1979). The Type II parameter *b*=6 is taken from Marmottant & Villermaux (2004a). The Type I, III, and IV parameters are chosen so that all size distributions have *R*=1.2. In all cases, *m*=3.



(a.) Log-log plane



(b.) Linear-linear plane

Figure 4. Gamma size distributions vs. large size spread test data for spray atomization taken from Turner & Moulton (1953). The Type II parameter b=2.81 is taken from Marmottant & Villermaux (2004b). The Type I, III, and IV parameters are chosen so that all size distributions have R=1.5525. In all cases, m=3.

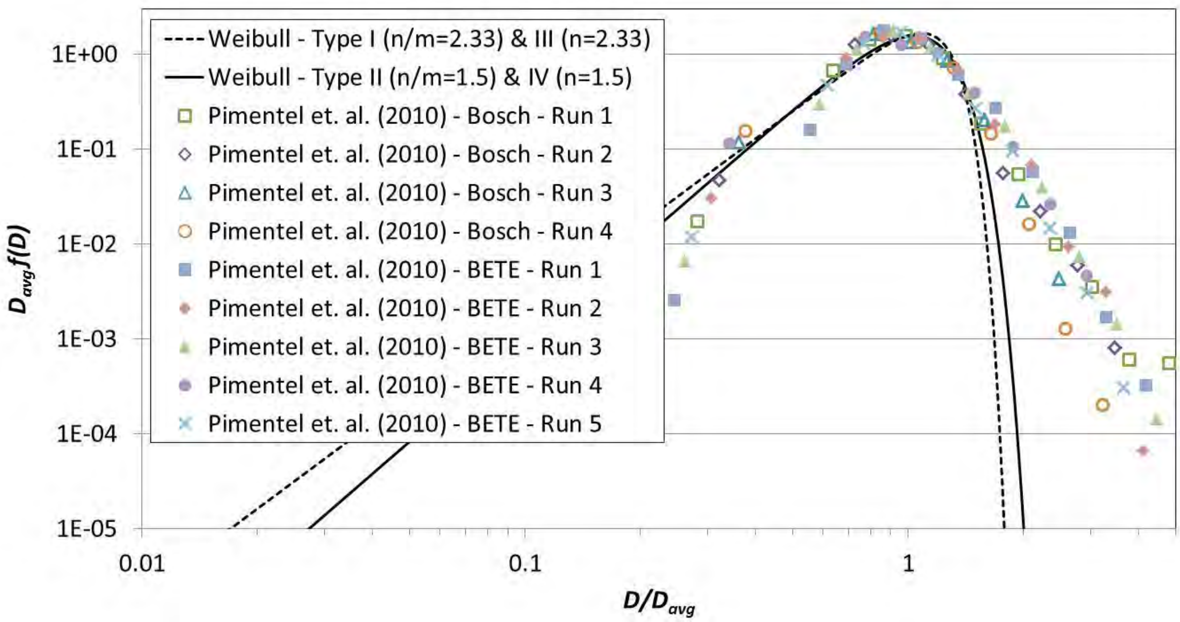
9.2 Weibull Size Distributions

Figures 5, 6, 7, and 8 compare Weibull size distributions to small, medium, and large size spread data, respectively. The Type II Weibull size distributions were taken from Onose & Fujiwara (2004), Grady et. al. (2001), and Mott & Linfoot (1943). The Type I, III, and IV Weibull size distributions were obtained by matching *R* to the Type II Weibull size distributions.

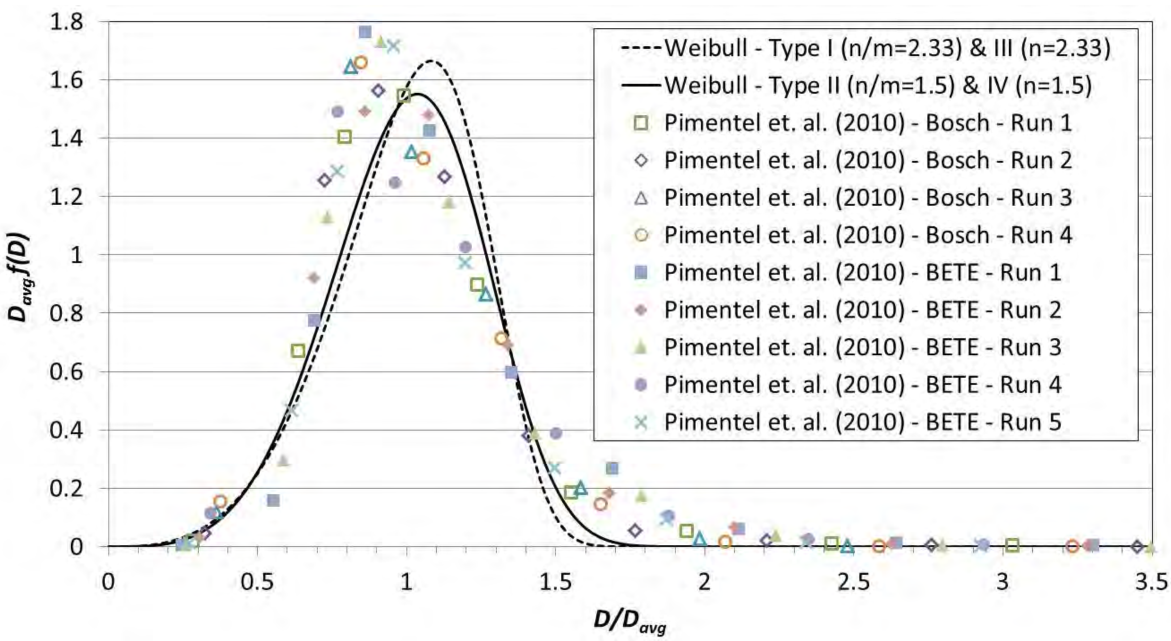
Near the mean for small size spreads, the Weibull distributions skew either right or left, while the test data is nearly symmetric. Other than this, Weibull size distributions offer a reasonably good match to test data. This is true despite the fact that the chosen Weibull distributions were originally meant for solid rather than liquid fragmentation.

Notice that positive Weibull distributions revert to a power law, i.e., they become linear in the log-log plane, for very small fragments. Similarly, negative Weibull distributions revert to a power law, i.e., they become linear in the log-log plane, for very large fragments.

As noted earlier, Type I and III Weibull distributions are identical to each other. Similarly, Type II and IV Weibull distributions are identical to each other. With the parameter choices made here, Type II and IV Weibull distributions tend to outperform Type I and III Weibull distributions.

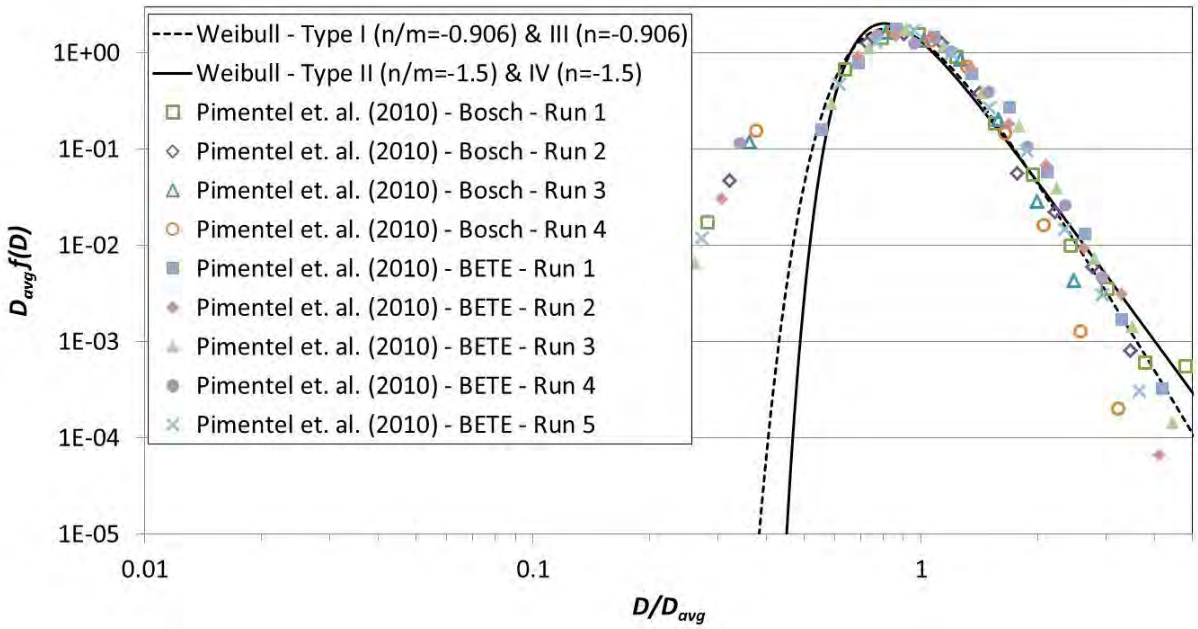


(a.) Log-log plane



(b.) Linear-linear plane

Figure 5. Weibull size distributions vs. small size spread test data for spray atomization taken Pimentel et. al. (2010). All size distributions have *R*=1.086 and *m*=3.



(a.) Log-log plane

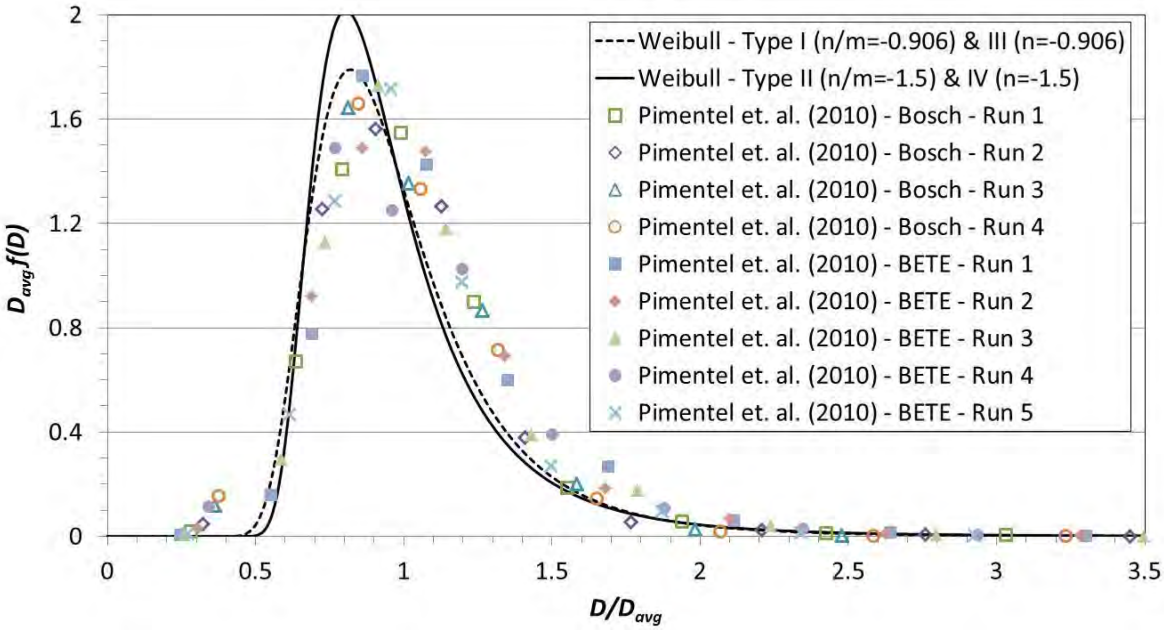
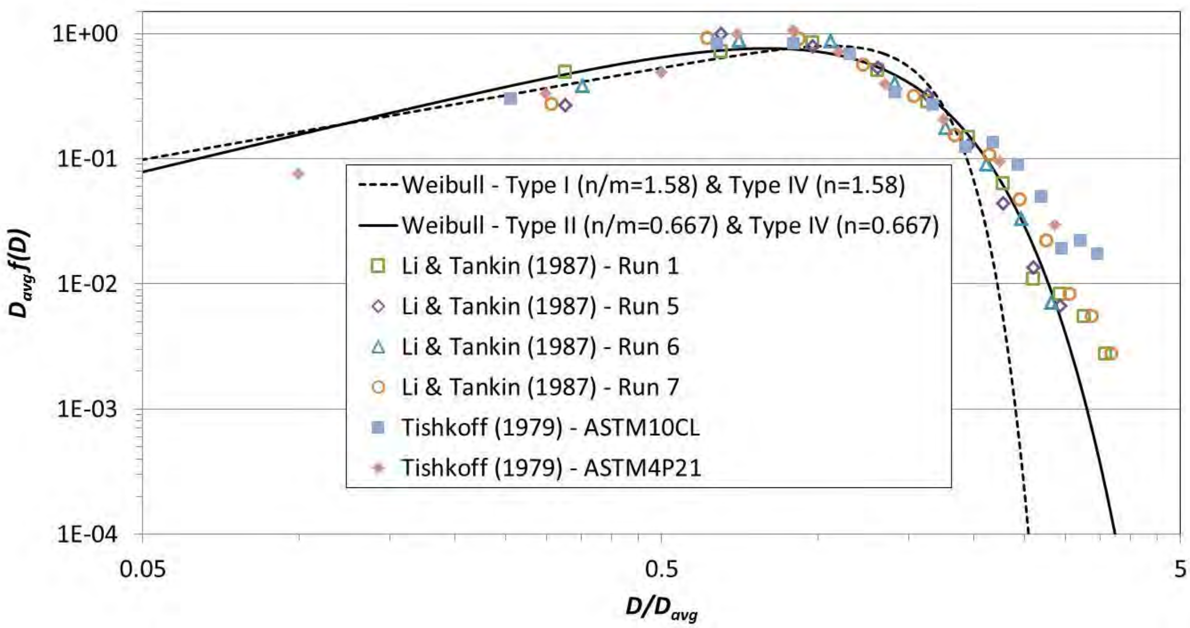
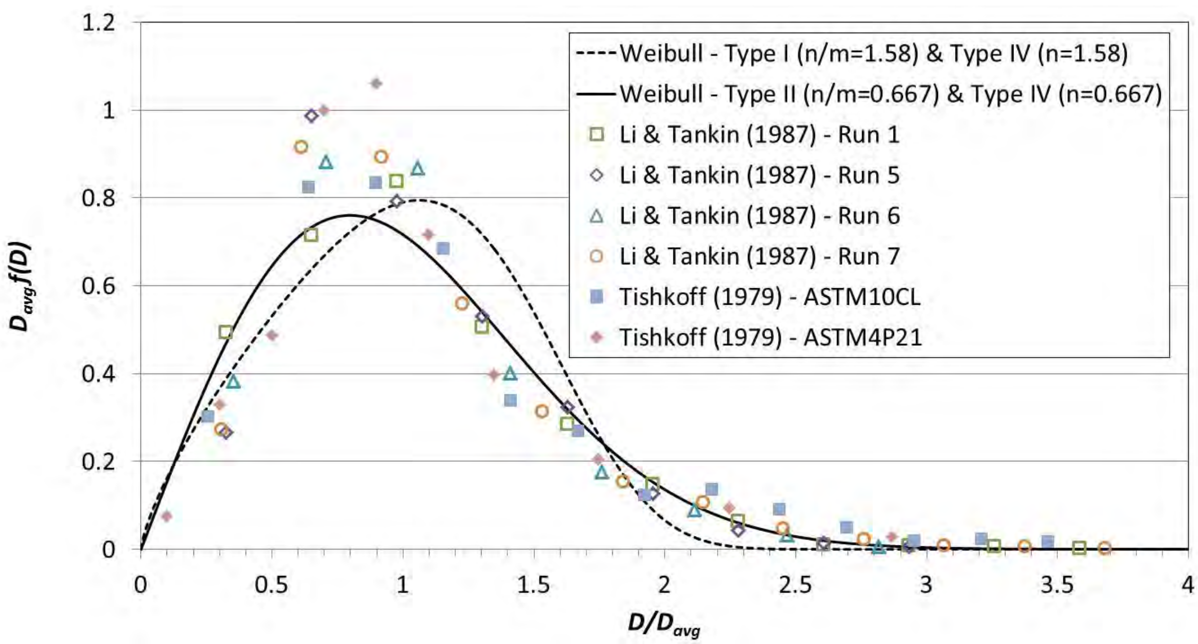


Figure 6. Negative (inverse) Weibull size distributions vs. small size spread test data for spray atomization taken Pimentel et. al. (2010). The Type II parameter *n/m*=-1.5 is taken from Onose & Fujiwara (2004). The Type I, III, and IV parameters are chosen so that all size distributions have *R*=1.086. In all cases, *m*=3.

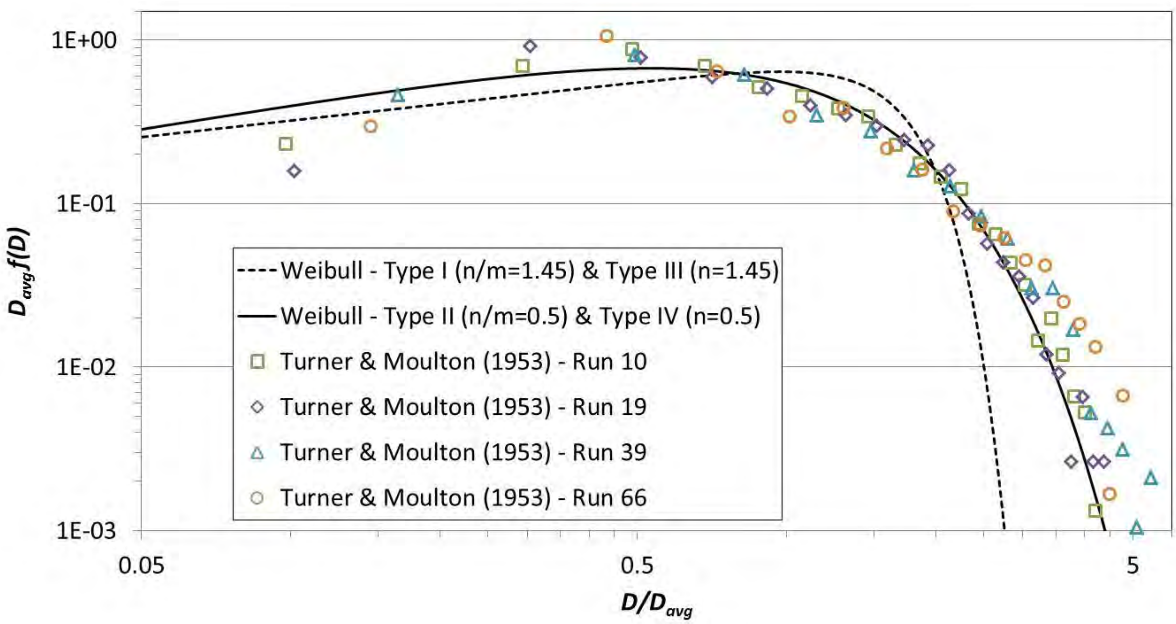


(a.) Log-log plane

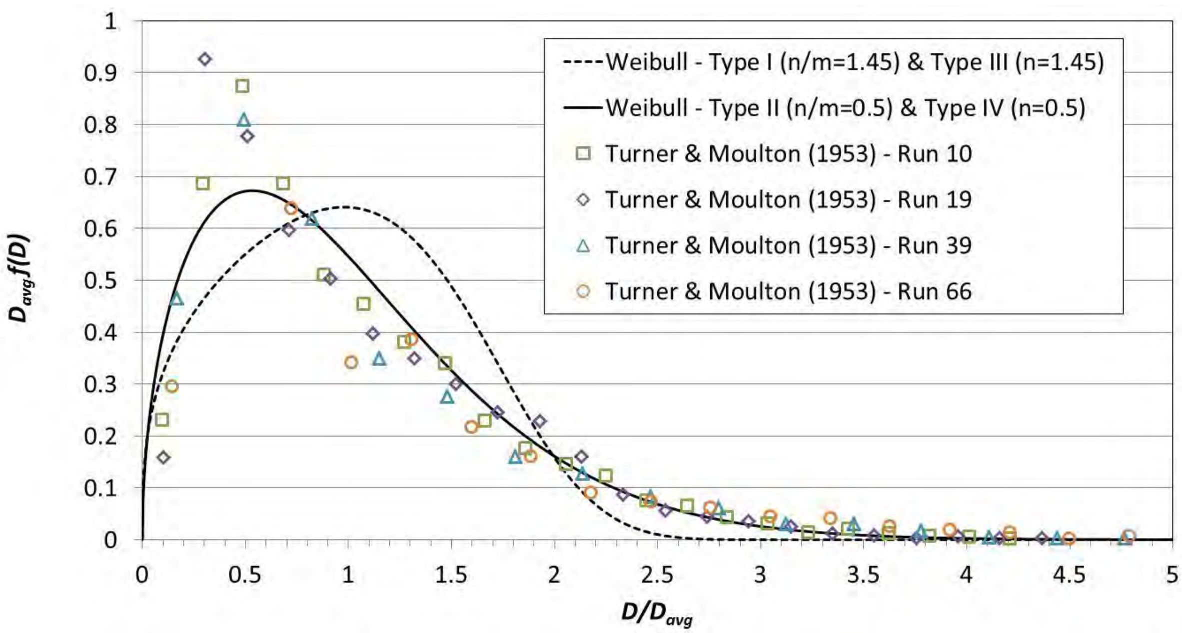


(b.) Linear-linear plane

Figure 7. Weibull size distributions vs. medium size spread test data for spray atomization taken from Li & Tankin (1987) and Tishkoff (1979). The Type II parameter *n/m*=0.667 is taken from Grady et. al. (2001). The Type I, III, and IV parameters are chosen so that all size distributions have *R*=1.5708. In all cases, *m*=3.



(a.) Log-log plane



(b.) Linear-linear plane

Figure 8. Weibull size distributions vs. large size spread test data for spray atomization taken from Turner & Moulton (1953). The Type II parameter *n/m*=0.5 is taken from Mott & Linfoot (1943). The Type I, III, and IV parameters are chosen so that all size distributions have *R*=2.418. In all cases, *m*=3.

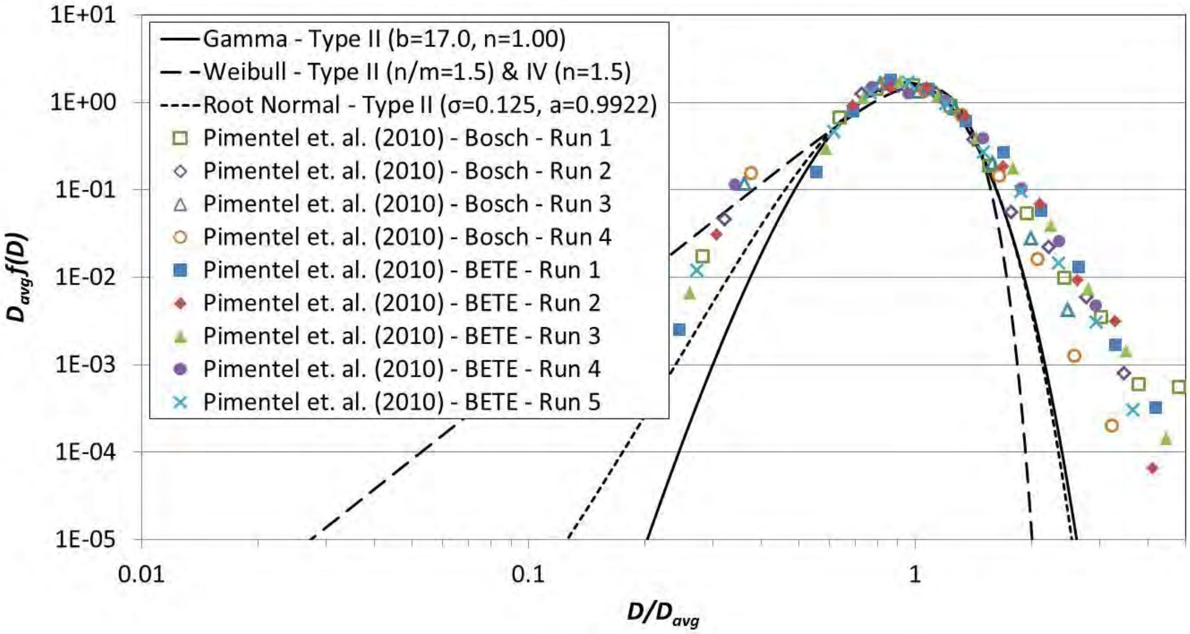
9.3 Gamma vs. Weibull vs. Root Normal Size Distributions

Figures 9, 10, and 11 compare Type II Gamma, Weibull, and root normal size distributions for small, medium, and large size spreads, respectively.

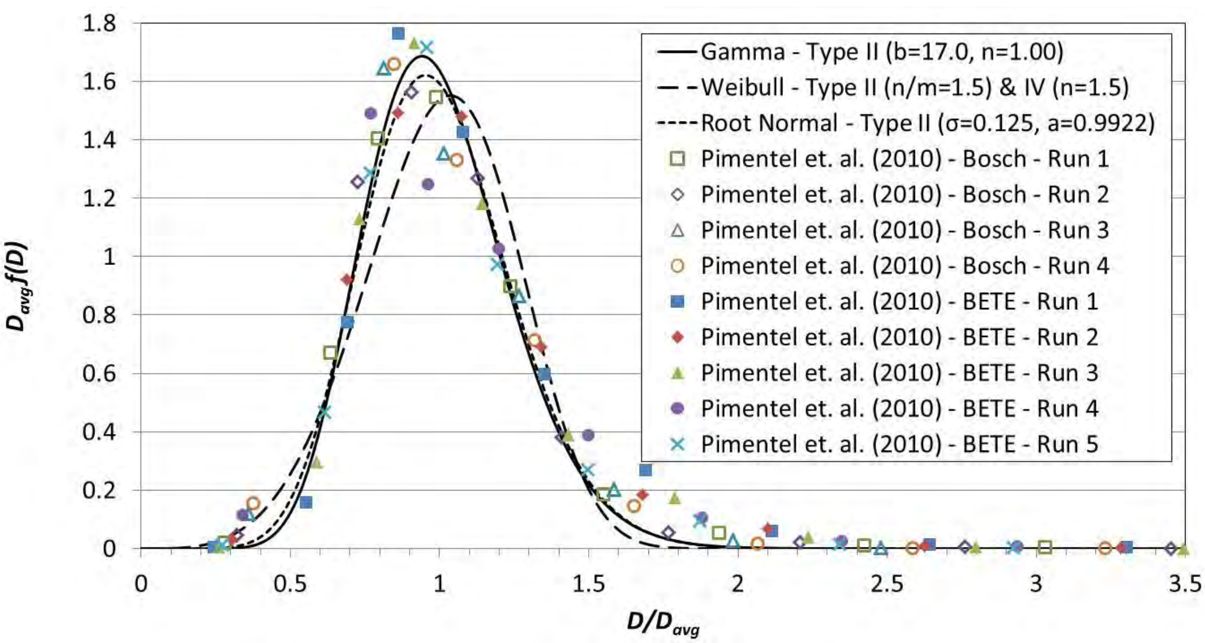
In Figure 9, the Gamma size distribution is taken from Villermaux (2007), the Weibull size distribution is taken from Onose & Fujiwara (2004), and the root normal size distribution is taken from Sallam et. al. (2006); see also Laney (2015). Notice that $R_M = 1.05 \pm 0.9\%$ in all three cases. Also notice that the Gamma and root normal size distributions are approximately the same, except for the smallest fragments.

In Figure 10, the Gamma size distribution is taken from Marmottant & Villermaux (2004a), the Weibull size distribution is taken from Grady et. al. (2001), and the root normal size distribution is taken from Wu et. al. (1991); see also Laney (2015). Notice that $R_M = 1.13 \pm 0.5\%$ in all three cases.

In Figure 11, the Gamma size distribution is taken from Marmottant & Villermaux (2004b), the Weibull size distribution is taken from Mott & Linfoot (1943), and the root normal size distribution is taken from Empie et. al. (1995, 1997); see also Laney (2015). Notice that $R_M = 1.20 \pm 0.7\%$ in all three cases. Also notice that the Weibull and root normal size distributions are approximately the same.

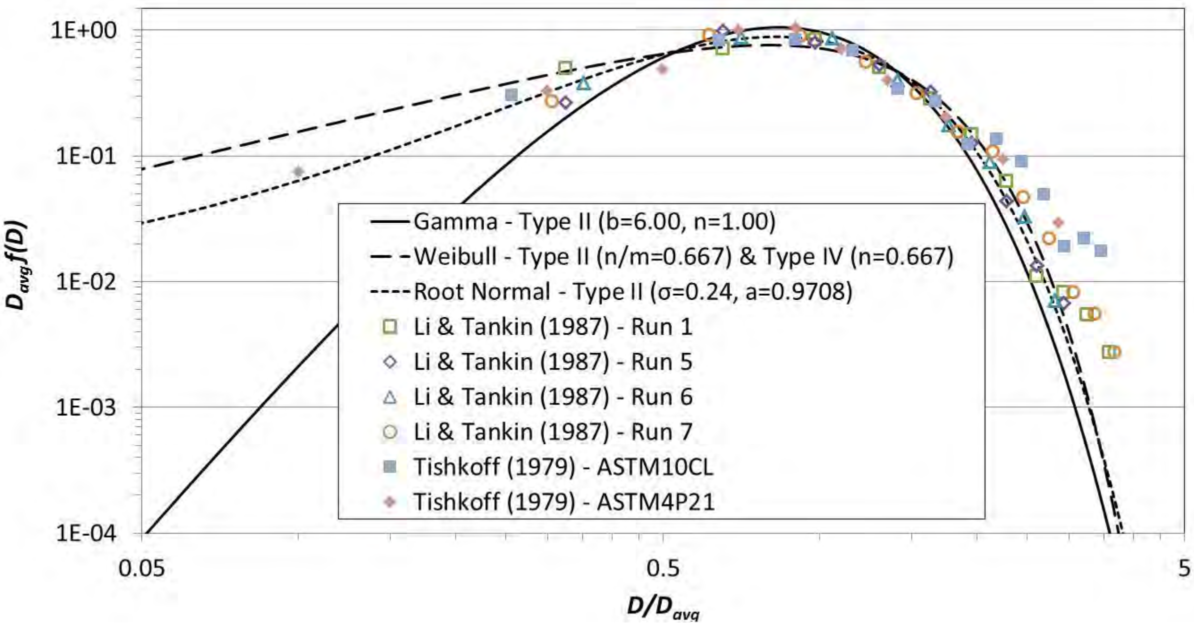


(a.) Log-log plane

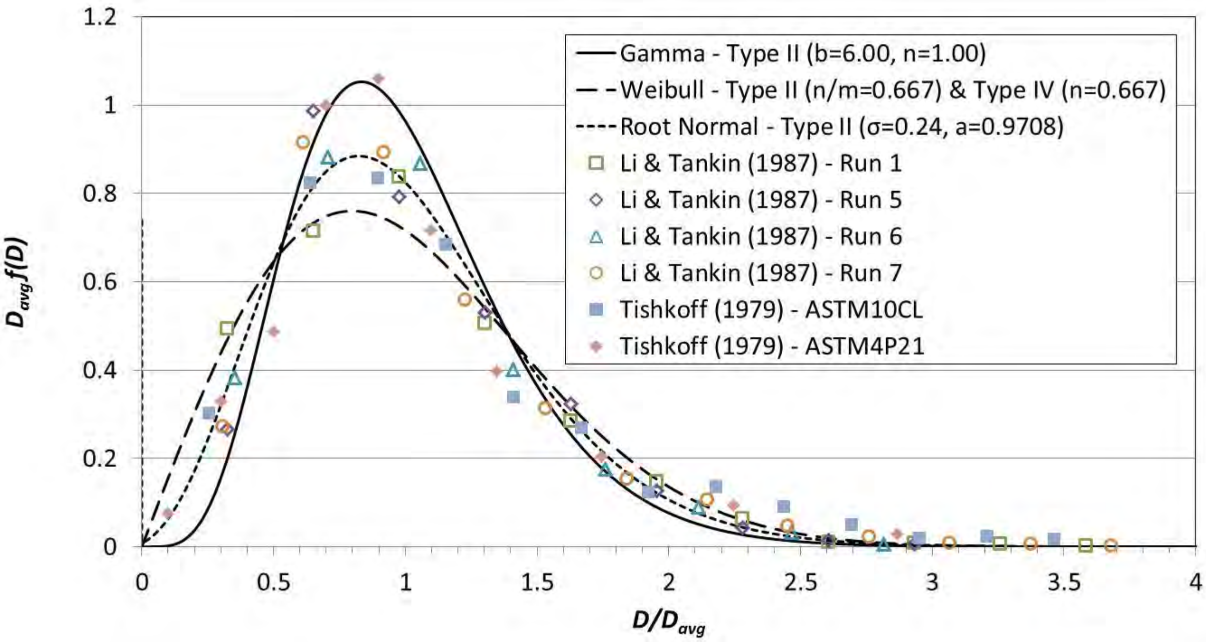


(b.) Linear-linear plane

Figure 9. Type II Gamma, Weibull, and root normal size distributions vs. small size spread test data for spray atomization taken from Pimentel et. al. (2010). All size distributions have $R_M \approx 1.05$ and m=3.

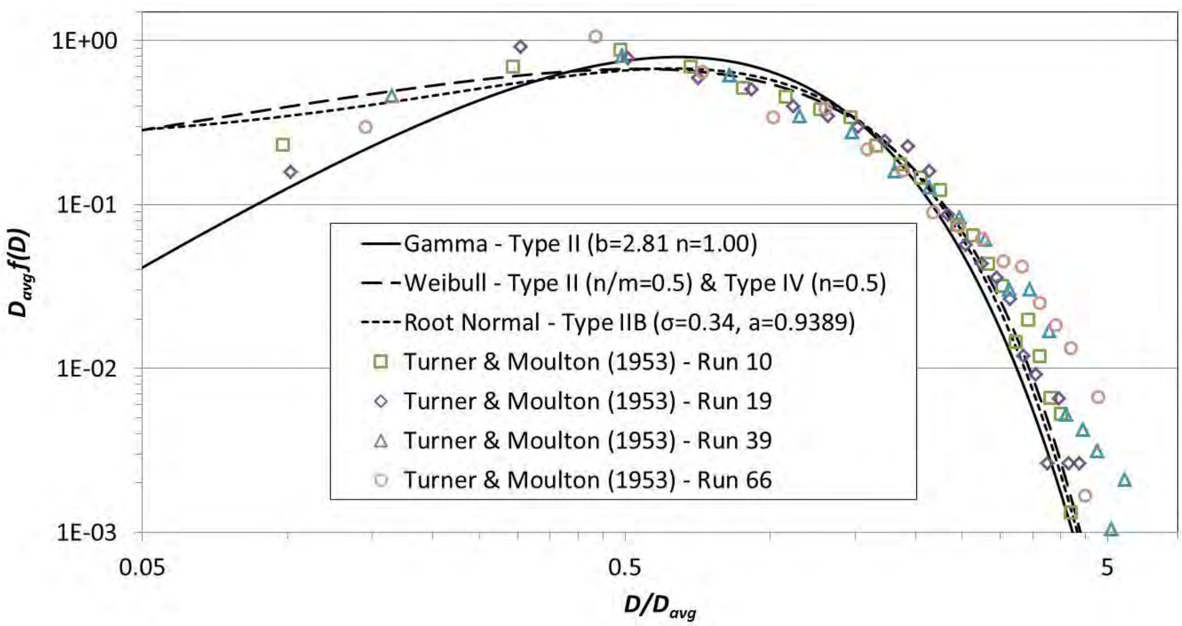


(a.) Log-log plane

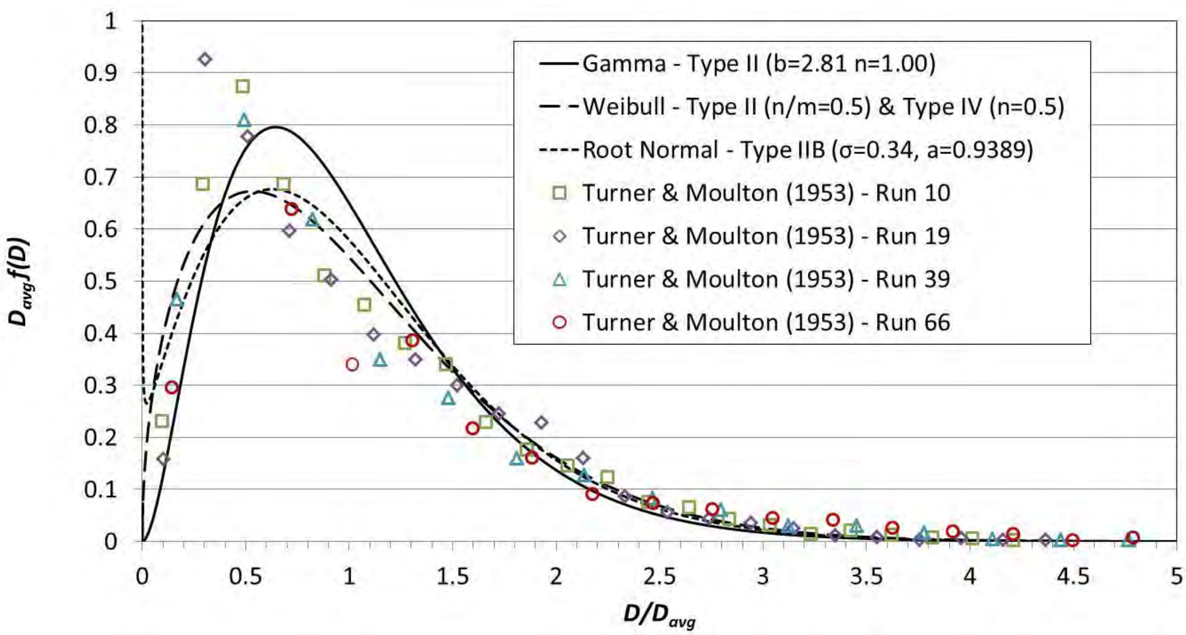


(b.) Linear-linear plane

Figure 10. Type II Gamma, Weibull, and root normal size distributions vs. medium size spread test data for spray atomization taken from Li & Tankin (1987) and Tishkoff (1979). All size distributions have $R_M \approx 1.13$ and m=3.



(a.) Log-log plane



(b.) Linear-linear plane

Figure 11. Type II Gamma, Weibull, and root normal size distributions vs. large size spread test data for spray atomization taken from Turner & Moulton (1953). All size distributions have $R_M \approx 1.20$ and m=3.

10. Conclusions

This paper has described a number of variants on traditional two-parameter Gamma and Weibull distributions. Each distribution has been classified as one of four types. Within a given type, each distribution has been categorized as either positive or negative, modified or unmodified.

For Weibull distributions, the traditional choices are Type I and Type II, unmodified and positive; all other variants described here are rare or new. For Gamma distributions, the traditional choices are Type II and IV, unmodified and positive; all other variants described here are rare or new. (Notice that there are no negative Gamma distributions.) While initial comparisons with test data are promising, more work is required to determine under what circumstances, if any, rare and new variants equal or outperform traditional choices.

Table 20 summarizes the transformation and self-similarity properties of Gamma and Weibull distributions. Notice that these properties are the same for all variants. For additional perspective, Table 20 also includes root normal size distributions; see Laney (2015).

Table 20. Summary of transformation and self-similarity conditions for three different families of size distributions.

Distribution	Transformation Condition *(No Severe Singularities Regardless of Form)*	Self-Similarity Conditions 1. *(Ensures Correct Average)*	2. *(Parameters Same for Different Averages)*	3. *(Distribution Same for Different Averages)*
Gamma *(all variants)*	✓	✓	✗	✓
Weibull *(all variants)*	✓	✓	✓	✓
Traditional Root Normal (Type I)	✗	✗	✗	✓
Improved Root Normal (Type II)	✓	✓	✗	✓

Gamma and root normal size distributions are traditionally used for liquid fragmentation while Weibull distributions are traditionally used for solid fragmentation. However, based on the comparisons given here, they both appear to be equally applicable to liquid and solid fragmentation. This is not surprising given that three-parameter Rosin-Rammler size distributions – which underlie Gamma and Weibull size distributions – are commonly used for both liquid and solid fragmentation.

Except for power laws, most if not all universal size distributions suggested in the research literature belong to one of the three families listed in Table 20. Of the families listed in Table 20, only Weibull size distributions obtain the second self-similarity condition. This means that the exact parameter choice is potentially meaningful for those universals, such as Mott-Linfoot and Marshall-Palmer, expressed as Weibull size distributions – but not for those expressed as Gamma or root normal size distributions.

Acknowledgements

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